Analysis of the radiated information in spinning sound fields[a]

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The information content of a spinning sound field is analyzed using a combination of exact and asymptotic results, in order to set limits on how accurately source identification can be carried out. Using a transformation of the circular source to an exactly equivalent set of line source modes, given by Chebyshev polynomials, it is found that the line source modes of order greater than the source wavenumber generate exponentially small fields. Asymptotic analysis shows that the remaining, lower order, modes radiate efficiently only into a region around the source plane, with this region shrinking as the mode order is increased. The results explain the ill-conditioning of source identification methods; the successful use of low order models in active noise control; and the low radiation efficiency of subsonic jets.

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Notation

c	speed of sound
I_m	radiated field of mth line source
J_m	Bessel function of first kind of order m
k	wavenumber
K	line source distribution
p	pressure amplitude
r	radial displacement
R	source-observer distance
s	line source coordinate
s_n	source amplitude
u_m	amplitude of line source mode
U_m	Chebyshev polynomial of the second kind of order m
z	axial displacement
β	$\cos^{-1} s$
θ	azimuthal angle
ω	radian frequency

I. INTRODUCTION

Source identification, the problem of determining an acoustic source from field measurements, has been attempted using a number of approaches in various different technologies. This paper examines the problem of identifying the source which generates a spinning acoustic field. The source which generates such fields can be represented as a set of modes which vary with azimuth on a circular disk, whether or not the source includes a spinning element. Examples include rotating systems such as cooling fans[1,2], helicopter rotors[3,4] duct terminations such as aircraft engine intakes[5–10] and jets[11] if a jet is modelled as a distribution of disk-shaped sources.

There are two broad categories of problem where source identification is required, corresponding to 'forward' and 'backward' projection of the field. In the forward problem, the aim is to estimate the source distribution accurately enough to allow the field to be predicted at positions other than the original measurement points. This has been done, for example, in extracting source parameters from near-field measurements of propeller noise, with the parameters then being used to calculate far-field noise[12].

In backward projection the problem is determination of the source proper from field measurements. This might be done in order to decide on noise control measures or because the acoustic source corresponds to some other physical variable of interest. In the first case, the aim is usually to find the acoustic source strength distributed over the source region, such as a rotor disk[1,2] or the termination of

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a duct[7–10]. An example of the second application is the study of noise generation by turbulent jets[11], where the aim is to determine the fluid-dynamical mechanisms which give rise to the acoustic source.

In any case, it is well known that the inverse problem, that of source identification, is ill-posed, meaning that small measurement errors can give rise to very large changes in the estimated source. Previous studies of the structure of spinning fields[13–16] have shown that the field decays exponentially away from the source. This means that in the forward projection problem, errors in the estimated acoustic source will decay and the predicted acoustic field may well be quite accurate, even if the source is not well recovered. On the other hand, in the backward projection problem, the exponential decay moving away from the source corresponds to exponential growth moving towards it, leading to large errors in the estimated source.

This problem can be framed in terms of subsonic and supersonic phase speed perturbations of the source. If we consider the model problem of a periodic source distribution $\exp[-\mathrm{j}k_x x]$ of temporal wavenumber $k = \omega/c$, distributed over an infinite plane $z = 0$, the radiated pressure at some distance z from the plane is[17]:

$$p \propto \frac{\exp \mathrm{j}(k^2 - k_x^2)^{1/2} z}{(k^2 - k_x^2)^{1/2}}.$$

This corresponds to a radiating field when $k > k_x$, i.e. the perturbations have supersonic phase speed. When $k < k_x$, the argument of the exponential is negative and the source does not generate a radiating field. In the case of the inverse problem, this means that a source distribution estimated from in-field measurements is band-limited to include only components with $k_x < k$. Such an analysis places a limit on the resolution with which a source distribution can be reconstructed. The same idea has also been used to estimate the part of a general source distribution which contributes to the radiated field, such as in the jet noise analysis of Freund[18].

In this paper, the radiated field from a finite disk source is analyzed to examine how much information about the source can actually be detected in the field. In a previous paper on a possible method for source identification[19], it was shown that the far-field noise is band-limited Fourier transform of a line source which is exactly equivalent to the disk source, similar to the spatial band-limiting for the infinite plane source. In this paper, without recourse to far field approximations, it is possible to establish fundamental limits on the number of degrees of freedom of a field, limits which are determined by the source frequency. The implications of the analysis are discussed with respect to some real source identification and radiation prediction problems.

II. ANALYSIS

The problem considered is that of the field generated by an azimuthally varying distribution of monopoles with strength $s(r_1, \theta_1) \exp[-\mathrm{j}\omega t]$ given by the Rayleigh integral[13,20]

$$p(r, \theta, z, \omega) = \int_0^1 \int_0^{2\pi} s(r_1, \theta_1) \frac{\mathrm{e}^{\mathrm{j}kR}}{4\pi R} \, \mathrm{d}\theta_1 \, r_1 \, \mathrm{d}r_1, \quad (1)$$
$$R^2 = r^2 + r_1^2 - 2rr_1 \cos(\theta - \theta_1) + z^2,$$

where the source is distributed over the unit disk in the plane $z = 0$, variables of integration have subscript 1 and the coordinate system is shown in Figure 1. The wavenumber $k = \omega/c$, and c is the speed of sound.

Taking one azimuthal mode of the source distribution, $s(r_1, \theta_1) = s_n(r_1) \exp \mathrm{j}n\theta_1$, the radiated field for one mode can be written $p = p_n \exp \mathrm{j}n\theta$:

$$p_n(r, z) = \int_0^1 \int_0^{2\pi} s_n(r_1) \frac{\mathrm{e}^{\mathrm{j}(kR - n\theta_1)}}{4\pi R} \, \mathrm{d}\theta_1 \, r_1 \, \mathrm{d}r_1, \quad (2)$$
$$R^2 = r^2 + r_1^2 - 2rr_1 \cos\theta_1 + z^2.$$

The integral of Equation 2 has been extensively studied due to its relevance to rotor acoustics and, under suitable conditions, as a good approximation to radiation from ducts. Many problems in source identification[1,2,5–10] can be viewed as attempts to recover the source term $s_n(r_1)$ from measurements of p_n.

The remainder of this paper consists of an analysis of the integral of Equation 2, which will establish limits on the information about the source which is radiated into the acoustic field, thereby fixing how accurately a source can be identified. The results are also applicable to the question of the detail with which a source need be specified in order to accurately predict the acoustic field, and to that of the radiation efficiency of jets.

A. Equivalent line source

The first step in the analysis is to transform the disk source into a line source which generates (exactly) the same acoustic field. This is a transformation which has been used, in the axisymmetric case,

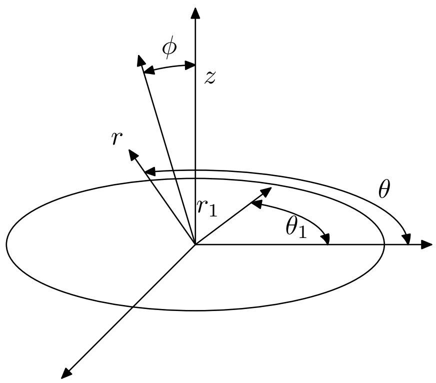

FIG. 1. Coordinate system for radiation prediction

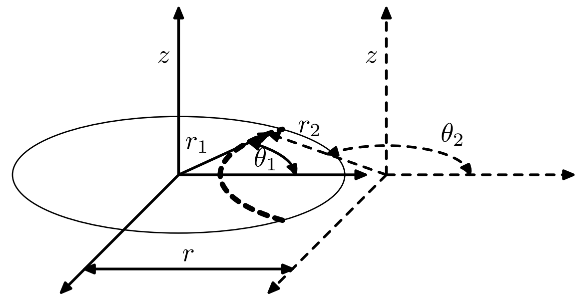

FIG. 2. Coordinates for transformation to equivalent line source

in studies of transient radiation from pistons[21,22], and, with azimuthal variation, in studies of rotor acoustics[13–16]. The first stage is to switch from the source-centred cylindrical coordinates (r, θ, z) of Figure 1 to the observer-centred coordinates (r_2, θ_2, z) of Figure 2. Under this transformation, Equation 2 becomes, for $r > 1$:

$$p_n(k, r, z) = \int_{r-1}^{r+1} \frac{\mathrm{e}^{\mathrm{j}kR}}{R} K(r, r_2) r_2 \,\mathrm{d}r_2, \tag{3}$$

$$R = \left(r_2^2 + z^2\right)^{1/2},$$

$$K(r, r_2) = \frac{1}{4\pi} \int_{\theta_2^{(0)}}^{2\pi-\theta_2^{(0)}} \mathrm{e}^{-\mathrm{j}n\theta_1} s_n(r_1) \,\mathrm{d}\theta_2, \tag{4}$$

where the source function $K(r, r_2)$ depends on r, the observer lateral separation, but is independent of z, the axial displacement. The coordinate systems are related by:

$$r_1^2 = r^2 + r_2^2 + 2rr_2 \cos\theta_2, \tag{5a}$$

$$\theta_1 = \tan^{-1} \frac{r_2 \sin\theta_2}{r + r_2 \cos\theta_2}. \tag{5b}$$

and the limits of integration in Equation 4 are given by setting $r_1 = 1$:

$$\theta_2^{(0)} = \cos^{-1} \frac{1 - r^2 - r_2^2}{2rr_2}. \tag{6}$$

The function $K(r, r_2)$ has square-root behavior at its end-points[19], $r_2 = r \pm 1$, and can be expanded:

$$K(r, r_2) = \sum_{m=0}^{\infty} u_m(r) U_m(s)(1 - s^2)^{1/2} \tag{7}$$

with $s = r_2 - r$ and $U_m(s)$ the Chebyshev polynomial of the second kind.

Inserting this expansion into Equation 3

$$p_n(k, r, z) = \sum_{m=0}^{\infty} u_m(r) I_m(k, r, z), \tag{8}$$

$$I_m(k, r, z) = \int_{-1}^{1} \frac{\mathrm{e}^{\mathrm{j}kR}}{R} U_m(s)(r + s)(1 - s^2)^{1/2} \,\mathrm{d}s, \tag{9}$$

$$= \int_{0}^{\pi} \frac{\mathrm{e}^{\mathrm{j}kR}}{R} (r + \cos\beta) \sin(m + 1)\beta \ \sin\beta \,\mathrm{d}\beta, \tag{10}$$

$$R^2 = (r + \cos\beta)^2 + z^2,$$

with the transformation $s = \cos\beta$ and use of the definition of the Chebyshev polynomial[23], $U_m(s) = \sin[(m + 1)\beta]/\sin\beta$.

III. RADIATED FIELD

The analysis of the previous section gives us a model of a spinning acoustic field expressed in terms of an exactly equivalent line source composed of a superposition of modes given as Chebyshev polynomials, with the modal coefficients functions of observer radius r, but not of axial displacement z. In this section, we use the model to draw basic conclusions about the acoustic information which is available for source identification.

A. Non-radiating modes

The first conclusion we can draw from the integral expression for I_m is that modes with $m > k$ generate fields of exponentially small amplitude and can be considered not to radiate. This is shown by evaluating $I_m(k, r, z)$ in the plane $z = 0$. This can be done exactly using Equation 10 and standard relations for Bessel functions[23]:

$$I_m(k, r, 0) = \mathrm{j}^m \pi(m+1) \frac{J_{m+1}(k)}{k} \mathrm{e}^{\mathrm{j}kr}. \quad (11)$$

In the case of large order $m+1$, with $k < m+1$, the asymptotic form of the Bessel function is, to leading order[23]:

$$J_{m+1}(k) \sim \frac{\exp[-(m+1)(\zeta - \tanh\zeta)]}{(2(m+1)\pi \tanh\zeta)^{1/2}}, \quad (12)$$

$$\zeta = \cosh^{-1} \frac{m+1}{k}. \quad (13)$$

Following the reasoning of Parry and Crighton[24], we note that since $\tanh\zeta < 1$ and ζ is large for small $k < m+1$, the argument of the exponential is large and negative, and the higher modes generate fields of exponentially small amplitude. Since $|I_m(r, z)|$ has its maximum in the plane $z = 0$, we can further conclude that the fields generated by modes with $m+1 > k$ are exponentially small everywhere and cannot be detected in the field. This is an exact result which places a first limit on the information radiated.

B. Radiating modes

A second limit on the information available in the acoustic field can be found by asymptotic analysis of I_m which can be rewritten:

$$\begin{aligned} I_m = (&Q_{m+2}(k, r, z) + Q_{-m-2}(k, r, z) \\ &- Q_m(k, r, z) - Q_{-m}(k, r, z))/4, \end{aligned}$$

with:

$$\begin{aligned} Q_m(k, r, z) &= \int_0^\pi \mathrm{e}^{\mathrm{j}k\psi(\beta)} \frac{r + \cos\beta}{R} \,\mathrm{d}\beta, \quad (14) \\ \psi(\beta) &= R + \gamma\beta, \\ \gamma &= m/k. \end{aligned}$$

The integral Q_m is in a suitable form for stationary phase analysis[25], which depends on finding the stationary points of ψ, i.e. values of β where $\mathrm{d}\psi/\mathrm{d}\beta = 0$ with $0 \leq \beta \leq \pi$. Upon differentiation and rearrangement, the condition $\mathrm{d}\psi/\mathrm{d}\beta = 0$ takes the form of a quartic equation:

$$(\alpha^2 - C^2)(r + C)^2 - \gamma^2 z^2 = 0, \quad (15)$$

where $C = \cos\beta$ and $\alpha^2 = 1 - \gamma^2$. To lie in the domain of integration, the stationary phase points must be real with $|C| < 1$. Such solutions exist only when $0 < \gamma < 1$ and for a limited range of z. At $z = 0$, the valid solutions for C are readily found as $C = \pm\alpha$. These values of C vary as $|z|$ increases until they coalesce into a conjugate pair at a value of z denoted z_c. The solutions $C_\pm$ are thus given by:

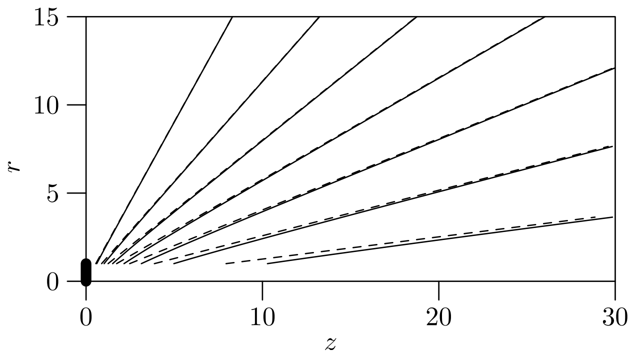

FIG. 3. Transition lines for $\gamma = m/8$, $m = 1, 2, \ldots, 6, 7$; exact solution solid; $z_c = \alpha r/\gamma$ dashed.

$$C_\pm(z) = \begin{cases} \pm\alpha, & z = 0, \\ -r/4 + (r^2 + 8\alpha^2)^{1/2}/4, & |z| = z_c. \end{cases} \quad (16)$$

where z_c, the value of z at which the stationary phase points merge is:

$$\begin{aligned} z_c &= (\alpha^2 - C_c^2)^{1/2}(r + C_c)/\gamma, \quad (17) \\ &\to \alpha r/\gamma, \quad \alpha/r \to 0. \end{aligned}$$

Written in spherical coordinates, the asymptotic form of $z_c = \alpha r/\gamma$ is a ray with polar angle $\phi = \sin^{-1}\gamma$. For completeness, we note that for $\gamma = 0$, there is no transition and the line source mode radiates into the whole field with amplitude proportional to $k^{-1/2}$.

Figure 3 shows the transition lines, exact and asymptotic, for $0 < \gamma < 1$. The accuracy of the asymptotic approximation for the transition line is confirmed and the plot shows which radiated modes are detectable in a given part of the acoustic field.

Using the previous results, the asymptotic behavior of the basic integral is given by:

$$\begin{aligned} Q_m \sim\, &\mathrm{j}^{3/2} \frac{\mathrm{e}^{\mathrm{j}k\psi_+}}{(kR_+)^{1/2}} \left[\frac{2\pi}{C_+(r + 2C_+) - \alpha^2}\right]^{1/2} (r + C_+) \\ &+ \mathrm{j}^{1/2} \frac{\mathrm{e}^{\mathrm{j}k\psi_-}}{(kR_-)^{1/2}} \left[\frac{2\pi}{C_-(r + 2C_-) - \alpha^2}\right]^{1/2} (r + C_-), \quad k \to \infty, \quad (18) \end{aligned}$$

$$R_\pm^2 = (r + C_\pm)^2 + z^2, \quad \psi_\pm = R_\pm + \gamma \cos^{-1} C_\pm,$$

and:

$$I_m \sim (Q_{m+2}(k,r,z) - Q_m(k,r,z))/4, \tag{19}$$

where Q_{-m} and Q_{-m-2} are neglected since they have no stationary phase points and decay much faster than Q_m and Q_{m+2} with increasing k.

Figure 4 compares the asymptotic approximation for $I_1(k,r,z)$ to a numerical evaluation of the integral. The transition value of z for $\gamma = 3/k$ is indicated and the real and imaginary parts of the integrals are plotted separately. As $z \rightarrow z_c$, the stationary phase approximation to Q_{m+2} breaks down and there is a resulting loss of accuracy. Away from this point, however, the approximation to I_m is accurate on both sides of z_c.

The asymptotic analysis shows that the radiating line source modes, those with $m < k$, radiate efficiently into a region bounded by $\pm z_c(r,\gamma)$. This gives a second limit on the information available in the acoustic field.

C. Far field approximation

For completeness, we give a far field approximation of the line source radiation integral, valid outside the region covered by the asymptotic expansions of §III.B. Using the standard approximation, $R \approx R_0 + (r - r_2)\sin\phi$, $1/R \approx 1/R_0$ with $R_0^2 = r^2 + z^2$:

$$I_m \approx \mathrm{j}^m \pi \frac{\mathrm{e}^{\mathrm{j}kR_0}}{R_0} \frac{m+1}{k\sin\phi} \left[\left(r + \mathrm{j}\frac{(m+2)}{k\sin\phi} \right) J_{m+1}(k\sin\phi) - \mathrm{j}J_m(k\sin\phi) \right] \tag{20}$$

so that in the far field, I_m decays as k^{-1}.

IV. INFORMATION IN SPINNING SOUND FIELDS

Summarizing the results of the previous section, the nature of a spinning sound field is seen to be determined by its wavenumber k and the relation of k to the set of modes contained in the equivalent line source. The first result, that line modes with $m > k$ generate exponentially small fields, means that the acoustic field contains a limited amount of information about the source. Such a result has been derived previously by showing that the far field pressure is a band-limited Fourier transform of the line source strength[19], but this new result establishes an exact limit on the information in the field, without needing a far-field approximation.

The second result, from the stationary phase analysis, shows that the modes which do radiate, those with $m < k$, are more efficient in some parts of the field than in others. The higher order radiating modes are detectable only near the source plane, with a lower radiation efficiency at larger z. The only mode which radiates efficiently over the whole field is the 'plane' mode $m = 0$.

The following sections discuss some implications of these findings for different problems.

A. Low speed rotors

One result which is of immediate interest is that, in some sense, low speed rotors have the same acoustic field. Given that for a system of radius a rotating at angular velocity Ω the non-dimensional wavenumber of the nth harmonic of the radiated field $k = n\Omega a/c = nM_\mathrm{t}$, with M_t the source tip Mach number, low speed rotors will have $k < 1$ over the first few harmonics of the signal. This means that the sound field is dominated by the zero order line mode and any set of rotors of a given blade number operating at the same speed, whatever their blade geometry, will have the same acoustic field, to within a scaling factor. Indeed, experiments in active noise control of noise from cooling fans[26] have found good results by discretizing the inverse model of the fan into three sections, i.e. using three degrees of freedom, for a value of $ka \approx 0.8$. The same group, in an earlier study of the conditioning of the inversion problem, using a hemispherical arrangement of microphones[1] found that the condition number reduced as frequency was increased, a finding they ascribe to "insufficient spatial resolution of the source for frequency below 200Hz." In the light of the analysis above, an alternative viewpoint might be that as frequency is increased, more line source modes radiate and the information available in the acoustic field becomes more nearly sufficient for source reconstruction.

B. Source identification

The original motivation for this work was the problem of identifying a rotating source. The results of §II and §III can be used to help show why this is a hard problem and to indicate how it might best be approached.

The first obvious consequence of the result of §III.A is that the acoustic field has a limited number of degrees of freedom. For a field of given wavenum-

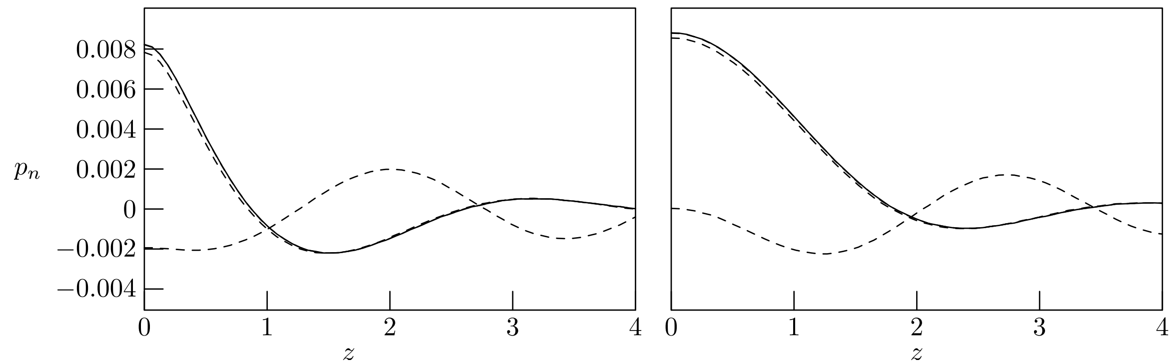

FIG. 6. Acoustic integral with exact line source (solid line) and lower order modes (dashed lines) for $m \leq 0, 2, 4$, $k = 2.5$, $r = 1.25$; real part on left, imaginary part on right.

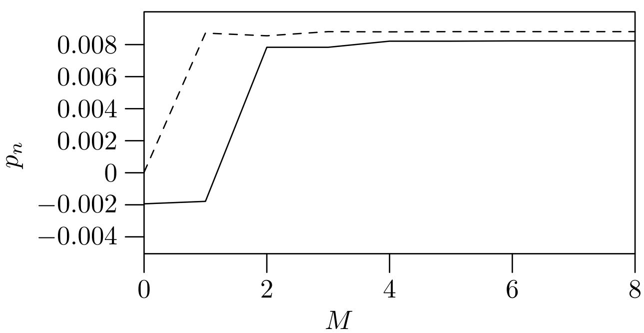

FIG. 7. Acoustic integral at $z = 0$ against maximum mode order M included: real part solid, imaginary part dashed.

Figure 7 shows the development of p_n at $z = 0$ as more modes are included in the summation. It is clear that the sum has all but converged after the $m = 2$ mode is added and is practically unchanged after the $m = 4$ mode is added, confirming the conclusion that the higher order modes do not radiate, in spite of their quite high amplitudes. The clear conclusion is that the acoustic field depends on the lower order modes and sources which differ only in the higher order terms of their line source decomposition cannot be distinguished by a source identification procedure. The corollary of this statement is that for an accurate prediction of the field, sources need only be specified to a resolution sufficient to correctly identify the amplitudes of the lower order modes.

D. Jet noise

In a recent paper[11], Jordan *et al.* examine noise production by a turbulent jet using proper orthogonal decomposition (POD) to perform a modal decomposition of the flow and a technique called MOD ('most observable decomposition') to decompose the acoustic far field. They find that while 350 flow modes are needed to account for 50% of the turbulent kinetic energy in the flow, only 24 modes are needed to account for 90% of the acoustic energy. If the jet is viewed as a distribution of disk sources along the jet axis, the results of this paper show that we should expect that only a small fraction of the modes will radiate noise and that in a complex source such as a jet, the bulk of the modes will have $m > k$ and will generate exponentially small fields. In a study of noise sources in a jet, Freund[18] filters the source terms to leave "a set of modes capable of radiating to the far field", based on a wavenumber criterion, but he notes that "additional cancellation may occur due to the radial structure of the source". The analysis of the previous sections offers an approach for the identification of the radial terms which will give such cancellation.

V. CONCLUSIONS

An analysis of the information content of a spinning sound field has been presented. It has been shown, on the basis of an exact analysis, that the acoustic field around a spinning source has at most M degrees of freedom with $M < k$, the acoustic wavenumber. This result arises from the replacement of the disk source with an exactly equivalent line source given by a sum of modes. Most of these modes generate exponentially small acoustic fields, i.e. do not radiate, and the remaining, lower order, modes radiate efficiently only into sectors of the acoustic field which become smaller as the modal order increases. The results explain a number of features which have been observed in the literature, including: the possibility of using low order source models for noise control; the ill-conditioning of source identification methods; and the low radiation efficiency of subsonic jets.

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