The classic theory for viscosity solutions is developed for proper E, as described by Crandall et al. [1992]. Neither the equilibrium equation (4) nor the travelling wave equation (6) are proper, since τ is not monotone. Second order degenerate elliptic equations which are not proper and defined on a bounded domain with Dirichlet data are discussed by Kawohl and Kutev [1999]. We now sketch a framework for non-proper second order degenerate equations on the real line. Here, it makes sense to define viscosity solutions (unlike for other non-proper equations, such as first order equations).

A *viscosity subsolution* to a second order degenerate elliptic equation

$$E(u, Du, D^2u) = 0 \quad \text{on } \mathbb{R} \tag{A.3}$$

is a function $u \in C^0(\mathbb{R})$ such that for every $v \in C^2(\mathbb{R})$ with $v(x_0) = u(x_0)$ and $v \geq u$ in a neighbourhood of x_0 it holds that

$$E(v(x_0), Dv(x_0), D^2v(x_0)) \leq 0. \tag{A.4}$$

Analogously, a *viscosity supersolution* to a second order degenerate elliptic equation (A.3) is a function $u \in C^0(\mathbb{R})$ such that for every $v \in C^2(\mathbb{R})$ with $v(x_0) = u(x_0)$ and $v \leq u$ in a neighbourhood of x_0 it holds that

$$E(v(x_0), Dv(x_0), D^2v(x_0)) \geq 0. \tag{A.5}$$

Finally, a *viscosity solution* to a second order degenerate elliptic equation is a solution which is both a sub- and a supersolution.

We remark that for proper E, with the same definition of a solution as above, various properties of solutions, such as a comparison principle, can be shown. For the equations considered here, while the concept of a viscosity solution is meaningful, several key properties do not hold. For example, there is *no* comparison principle, as shown in Section 3.

We remark that viscosity solutions to the travelling wave equation (6) are also viscosity solutions to the governing equation (1). We first write parabolic questions, such as (1), as degenerate elliptic equations in the variable $y := (x, t)$. To see that the viscosity solutions of Proposition 4.2 are viscosity solutions of the original equation, one needs to consider functions $v = v(x, t)$ with $v(x, t) \geq \phi(x - ct)$ (and $v(x, t) \leq \phi(x - ct)$) and $v(x_0, t_0) = \phi(x_0 - ct_0)$ and show that for

$$E(v, Dv, D^2v) := v_t - F(v_x)\left[\epsilon v_{xx} + \tau(g - v)\right],$$

it holds that

$$E(v(x_0, t_0), Dv(x_0, t_0), D^2v(x_0, t_0)) \leq E(\phi(x_0, t_0), D\phi(x_0, t_0), D^2\phi(x_0, t_0)),$$