

# On the fully-nonlinear shallow-water generalized Serre equations

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(Dated: December 17, 2009)

A fully-nonlinear weakly dispersive system for the shallow water wave regime is presented. In the simplest case the model was first derived by Serre in 1953 and rederived various times since then. Two additions to this system are considered: the effect of surface tension, and that of using a different reference fluid level to describe the velocity field. It is shown how the system can be further expanded by consistent exchanges of spatial and time derivatives. Properties of the solitary waves of the resulting system as well as a symmetric splitting of the equations based on the Riemann invariants of the hyperbolic shallow water system are presented. The latter leads to a fully-nonlinear one-way model and, upon further approximations, existing weakly nonlinear models. Our study also helps clarify the differences or similarities between existing models.

## I. INTRODUCTION

There are a wealth of shallow water wave models in the literature. These fall in certain categories: fully- or weakly-nonlinear, nondispersive or weakly-dispersive and unidirectional or bidirectional (or for a two-dimensional free-surface they can be weakly two-dimensional or isotropic). In addition, models are either based on an a priori assumption of irrotationality, using a velocity potential, or using the primitive velocity variables - with irrotationality sometimes as

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a “hidden” assumption. In [1] one finds a recent review of the interrelation of these models. In applications, one also often seeks to include topographical effects in these models.

We present here a generalization of the Serre equations [2]. These are fully-nonlinear, weakly dispersive, bidirectional (or isotropic) equations that have a built-in assumption of irrotationality. They are

$$h_t + (hu)_x = \mu^2 \mathcal{Y}_m \quad (1)$$

$$(hu)_t + \left( hu^2 + \frac{h^2}{2} \right)_x = \mu^2 (\mathcal{Y}_p + \mathcal{D} + \mathcal{S}) \quad (2)$$

where

$$\begin{aligned} \mathcal{Y}_m &= -(\alpha^2 - 1) \frac{1}{6} [h^3 u_{xx}]_x \\ \mathcal{Y}_p &= -(\alpha^2 - 1) \left( \frac{1}{6} [h^3 u_{xx}]_t + \frac{1}{3} [h^3 uu_{xx}]_x \right) \\ \mathcal{D} &= \left[ \frac{1}{3} [u_{xt} + uu_{xx} - u_x^2] h^3 \right]_x \\ \mathcal{S} &= -B \left[ \frac{1}{2} h_x^2 - hh_{xx} \right]_x \end{aligned}$$

Henceforth we shall denote this system gSerre. For simplicity we have written these equations for a one-dimensional free-surface and without including the effect of variable bathymetry. Both of these effects are easily included. The dimensional depth of the fluid is given by  $h_0 h(x, t)$  and  $\sqrt{gh_0} u(x, t)$  is the dimensional velocity at depth  $y = \alpha h / \sqrt{3}$ . The equations have three parameters:  $0 \leq \mu \ll 1$  is the long-wave, or dispersion parameter: the ratio of the depth to the typical horizontal scale of motion and also the order of the velocity variation in the vertical;  $0 \leq B < \infty$  is the Bond number ( $B = \gamma / gh_0^2$ ), the ratio of the length scale of surface tension effects to the depth of the fluid. Selecting  $0 \leq \alpha \leq \sqrt{3}$  guarantees that one is measuring a velocity inside the fluid. As we shall see below, it is reasonable to restrict further  $0 \leq \alpha \leq 1$  so as to ensure well-posedness of the linearized problem. The value  $\alpha = 1$  is special since it also corresponds to the equations written for the vertical mean of the horizontal velocity. In that case  $\mathcal{Y}_m = \mathcal{Y}_p = 0$  and the first equation is exact to all orders in  $\mu^2$ .

The history of these equations is interesting: the simplest case when the right-hand side is zero ( $\mu = 0$ ) are the hyperbolic “shallow-water” or St. Venant system first written in 1871 [3]. The system derived by Serre in 1953 is obtained by setting all but  $\mathcal{D}$  equal to zero ( $\alpha^2 = 1, B = 0$ ). Either the Serre system or equivalent equations have been rederived in the literature many times, notably by Su and Gardner [4] in 1969, and, using different arguments, by Green and Naghdi [5] in 1976. In [6] a numerical method is based on a potential flow derivation equivalent to these equations with  $B = 0$ . All weakly-nonlinear equations of order  $\mu^2$  in dispersion can be obtained from

further approximations and manipulations of (1) and (2). These include a variety of Boussinesq equations, and the unidirectional Korteweg-de Vries and Camassa-Holm equations [7].

Numerically, the Serre and gSerre systems have the difficulty of having a time derivative in the nonlinearity. This problem either can be faced directly, which involves the inversion of a full matrix at each timestep, or one can seek alternative equations, formally of the same order of approximation, that do not have this problem. A naive approach, the substitution of the leading order St. Venant dynamics to remove the time derivatives in the terms of order  $\mu^2$  of the original Serre equations, fails since it yields a linearly ill-posed system. However, as shown below, with the inclusion of the parameter  $\alpha$  it is possible to circumvent this and write a “time-explicit”, consistent, gSerre model. We also show another alternative that circumvents the time derivative in the nonlinearity with a fully nonlinear one-way model.

## II. SOLITARY WAVES OF THE GSERRE SYSTEM

The gSerre system, linearized about  $u = 0$  and  $h = 1$  has the dispersion relation

$$c^2 \equiv \left(\frac{\omega}{k}\right)^2 = \frac{[1 + \mu^2 \frac{1}{6}(1 - \alpha^2)k^2][1 + \mu^2 Bk^2]}{1 + \mu^2 \frac{1}{6}(3 - \alpha^2)k^2}$$

from which we deduce that in order to have well-posedness of the linear system, we need  $0 \leq \alpha^2 \leq 1$ . The gSerre system, under the traveling wave assumption which allows one integration of each equation, is a second-order system for either  $B = 0$  or  $\alpha = 1$ , or a fourth-order system otherwise. In all cases there are periodic and solitary wave solutions. We shall focus here on the latter.

With  $\alpha = 1$ ,  $B = 0$ , the system (1) has the closed form solitary waves [2, 10]

$$\begin{aligned} \bar{h} &= 1 + A \operatorname{sech}^2[\mu K(x - ct)], & \bar{u} &= c \frac{\bar{h} - 1}{\bar{h}} \\ c^2 &= 1 + A, & (\mu K)^2 &= \frac{3}{4} \frac{A}{(1 + A)}. \end{aligned}$$

For  $\alpha = 1$ ,  $B \neq 0$ , the solitary waves are not expressed in closed form but satisfy the differential equation

$$\bar{h}'^2 = \frac{(\bar{h} - 1)^2(\bar{h} - c^2)}{B\bar{h} - \frac{c^2}{3}},$$

and thus, their speed amplitude relation is still given by  $c^2 = h_m$ , where  $h_m$  is the fluid depth at the crest or trough of the wave. It is easy to see that if  $B < 1/3$  the solitary waves are of elevation and they can be of arbitrary amplitude, whereas for  $B > 1/3$  they are of depression. This is an extension to fully nonlinear waves of a result known in

the weakly nonlinear regime. Furthermore, depression waves are limited in amplitude by  $h_m = 0$  where there is a stationary flow with a contact point with, locally,  $h \sim |x|/\sqrt{B}$ .

For  $0 \leq \alpha < 1$ , the solitary waves still exist, but they cannot be written in closed form, even for  $B = 0$ . In this parameter regime, there is a limiting largest wave. Contrary to the Camassa-Holm equation where, for certain parameter values, the wave of limiting amplitude has a corner, the limiting waves of gSerre are smooth at their peaks. Speed-amplitude curves and wave profiles are shown in Figures 1 and 2. The values  $\alpha^2 = 0.66, 0.15$  were chosen in the figures since the former corresponds to an optimal parameter for the linear dispersion characteristics of the problem [11] and the latter gives a limiting wave whose speed and amplitude correspond approximately with that of the Euler equations.

Turning to the solutions with  $B \neq 0$ , Figure 3 shows solitary wave solutions for  $\alpha = 1$ . The effect of capillarity makes an elevation wave ( $B < 1/3$ ) narrower than a pure gravity wave of equal speed, and for depression waves ( $B > 1/3$ ), the trough becomes sharper as the amplitude increases. For  $\alpha \neq 1$ , the depression waves change little, arising from the fact that the velocity profile becomes more uniform as the fluid becomes shallower under the wave. The elevation waves, however, are qualitatively different in the far field with  $\alpha \neq 1$ : since  $k = 0$  is no longer the global maximum of  $c$  one expects and observes generalized solitary waves bifurcating from  $k = 0$  and wavepacket solitary waves bifurcating from the global minimum of  $c$  at finite  $k$ . This nonmonotonic behavior of  $c(k)$  is similar to that for the full Euler equations for gravity-capillary waves with  $B < 1/3$  and one can thus use  $\alpha$  as a tuning parameter to approximate the Euler dispersion relation.

### III. CONSISTENT MODIFICATIONS OF THE GSERRE SYSTEM

In this section we propose formally consistent modifications of the gSerre system that can be used to introduce additional properties to the equations. The idea has been used before to yield multi-parameter families of models such as the three-parameter family of Boussinesq equations presented in [9]. Here, we may consider substituting, in the order  $\mu^2$  terms of the gSerre equations

$$\begin{aligned} h_t &\rightarrow (1 - \beta)h_t - \beta(hu)_x \\ u_t &\rightarrow (1 - \delta)u_t - \delta(uu_x + h_x), \end{aligned}$$

This leads to a three-parameter  $(\alpha, \beta, \delta)$  family of models. We shall consider here as an example the case  $\beta = \delta = 1$  which results in removing completely the time derivatives from the nonlinear terms

$$h_t + (hu)_x = \mu^2 \mathcal{Y}_m \quad (3)$$

$$(hu)_t + \left(hu^2 + \frac{h^2}{2}\right)_x = \mu^2 (\overline{\mathcal{Y}}_p + \overline{\mathcal{D}} + \mathcal{S}) \quad (4)$$

where  $\mathcal{Y}_m$  and  $\mathcal{S}$  are unchanged and

$$\begin{aligned} \overline{\mathcal{Y}}_p &= -(\alpha^2 - 1) \left( \frac{1}{6} [h^3 u u_{xx}]_x - \frac{5}{6} h^3 u_x u_{xx} - \frac{1}{6} h^3 h_{xxx} \right) \\ \overline{\mathcal{D}} &= - \left[ \frac{1}{3} [2u_x^2 + h_{xx}] h^3 \right]_x \end{aligned}$$

In order for this model to be linearly well posed, it requires  $\alpha^2 = 2$  if  $B = 0$  (in the weakly nonlinear case this corresponds to the KdV-KdV version of the Boussinesq system [9]). Figures 4 and 5 present typical computations for initial value problems of the above system with and without the effect of surface tension.

#### IV. SYMMETRIC SPLITTING AND ONE-WAY MODELS

A common goal is to write model equations for unidirectional wave propagation. For the shallow water equations ( $\mu^2 = 0$ ) this is accomplished in terms of Riemann invariants [12]. Define

$$R^\pm = \frac{u}{2} \pm (\sqrt{h} - 1), \quad c^\pm = u \pm \sqrt{h},$$

where we have subtracted the mean depth from the usual Riemann invariants. One can invert these relationships and write

$$u = R^+ + R^-, \quad \sqrt{h} = 1 + \frac{1}{2}(R^+ - R^-),$$

and express the characteristic speeds of the system in terms of the Riemann invariants

$$c^\pm = u \pm \sqrt{h} = \frac{3}{2}R^\pm + \frac{1}{2}R^\mp \pm 1.$$

In this way the shallow water system can be written as

$$\begin{aligned} R_t^+ + \left( \frac{3}{2}R^+ + \frac{1}{2}R^- + 1 \right) R_x^+ &= 0 \\ R_t^- + \left( \frac{3}{2}R^- + \frac{1}{2}R^+ - 1 \right) R_x^- &= 0 \end{aligned}$$

Taking one Riemann invariant equal to zero is consistent, and indicates an exact decoupling, in these variables, of left- and right- traveling waves. Unfortunately, no exact decoupling is known for more complicated water wave models and Riemann invariants apply only to systems of two first order equations. For equations with higher derivatives, such as the Boussinesq equation, the splitting that leads to the Korteweg-de Vries equation only decouples left- and right-modes approximately in the small amplitude, small dispersion limit. Here, using the Riemann invariants as a basis, we write an approximate one-way system, from the gSerre equations, that does not rely on small amplitude, only on small dispersion. A symmetric splitting based on the shallow water Riemann invariants for the gSerre system (1) and (2) yields

$$R_t^\pm + c^\pm R_x^\pm = \mu^2 \mathcal{N}^\pm(R^+, R^-), \quad (5)$$

where  $\mathcal{N}^\pm = (\alpha^2 - 1)\mathcal{M}^\pm + \mathcal{H} + \mathcal{T}$  with

$$\begin{aligned} \mathcal{M}^\pm &= -\frac{1}{12} \left[ (h^2 u_{xx})_t + c^\pm (h^2 u_{xx})_x + 2h^2 u_{xx} R_x^\pm \right], \\ \mathcal{H} &= \frac{1}{6h} \left[ (u_{xt} + (2 - \alpha^2)uu_{xx} - (u_x)^2) h^3 \right]_x, \\ \mathcal{T} &= -\frac{B}{2h} \left[ \frac{1}{2} h_x^2 - hh_{xx} \right]_x, \end{aligned}$$

and all terms can be expressed in terms of the Riemann variables. Note that the nonlinearity in the two equations is the same if  $\alpha^2 = 1$ . The system (5) is completely equivalent to (1) and (2), but in these symmetrized variables, the two equations are only weakly coupled: if, initially, one of the Riemann variables is zero (say  $R^-$ ) it will not remain zero, but will remain of order  $\mu^2$ . Thus, writing  $R^- = \mu^2 L$ ,  $R^+ = R$ , a formally consistent truncated system for one way waves is

$$\begin{aligned} R_t + \left( \frac{3}{2}R + 1 \right) R_x &= \mu^2 \left[ \mathcal{N}^+(R, 0) - \frac{1}{2}LR_x \right], \\ L_t + \left( \frac{1}{2}R - 1 \right) L_x &= \mathcal{N}^-(R, 0). \end{aligned} \quad (6)$$

Although this appears as a “two-way” system, one can (implicitly) solve for  $L$  given  $R$  in the second equation and conclude that  $L$  provides only a nonlocal correction, through the  $\mu^2 LR_x$  term, to the characteristic speed of the right-traveling wave. Of course, the above system is consistent only if  $L$  is initially small. We have obtained a computationally simple, strongly nonlinear system approximating weakly dispersive one-way waves. The only additional approximation from the Serre system is that one of the Riemann variables is small. To illustrate, consider now the simplest case where  $\alpha^2 = 1, B = 0$  and where the leading order behavior of  $R$  is used to remove time

derivatives in the nonlinear terms. This results in  $\mathcal{N}^+(= \mathcal{N}^-)$  given by

$$-\frac{1}{6h} \left[ \left( R_{xx} + \frac{1}{2} R R_{xx} + \frac{5}{2} (R_x)^2 \right) h^3 \right]_x. \quad (7)$$

Figure 6 shows the breakup on an initial disturbance into right traveling waves for this system. Note that, consistently,  $L$  remains small throughout the evolution. In fact, we propose the Riemann invariant formulation as an effective method to initialize strongly nonlinear shallow water two-way equations with one-way data. One can test the effectiveness of this method by applying it to the system derived in section III. Figure 7 shows the result of a fully bidirectional model initialized for one way propagation with  $R^- = 0$ .

## V. WEAKLY NONLINEAR MODELS

We now show that the addition of a small amplitude assumption in (6) results in known *weakly nonlinear* one-way equations of the KdV-BBM type with nonlinear dispersive corrections. Scaling  $R$  and  $L$  with an independent small parameter  $\epsilon$ , with  $1 \gg \epsilon \gg \mu^2$ , and for simplicity taking again  $\alpha^2 = 1$ ,  $B = 0$ , we can write

$$\begin{aligned} R_t + R_x &= -\epsilon \frac{3}{2} R R_x + \mu^2 \frac{1}{6} R_{xxt} + \epsilon \mu^2 \left[ \frac{1}{6} R R_{xxx} - \frac{1}{6} R_x R_{xx} + \frac{1}{3} R R_{xxt} + \frac{1}{2} R_x R_{xt} - \frac{1}{2} L R_x \right], \\ L_t - L_x &= \frac{1}{6} R_{xxt}. \end{aligned}$$

Here, contrary to Section IV, it is simple to express  $L$  in terms of  $R$ . For example, we can take  $L \approx \frac{1}{12} R_{xx}$  and exchanging time-derivatives for spatial ones in the nonlinear terms, one obtains immediately the BBM equation with further nonlinear dispersive corrections

$$R_t + R_x + \epsilon \frac{3}{2} R R_x - \mu^2 \frac{1}{6} R_{xxt} + \epsilon \mu^2 \left[ \frac{1}{6} R R_{xxx} + \frac{17}{24} R_x R_{xx} \right] = 0. \quad (8)$$

Similar equations of the KdV-BBM type have been derived, most notably the Camassa-Holm equation which differs from the one above in the coefficients on the  $\epsilon \mu^2$  terms. However, the Camassa-Holm coefficients can be obtained with a particular value of  $\alpha^2$  [13].

## VI. CONCLUSIONS

The gSerre system presented above is a generalization of other fully-nonlinear shallow-water systems that have been derived independently in the literature often without the realization that they were equivalent. We have included the effects of surface tension, of expanding the velocity field about an arbitrary level, and proposed further modifications

using a consistent exchange between time- and spatial derivatives in the dispersive terms. We believe that this system warrants further mathematical study and, together with the addition of topographical effects could prove useful in near-shore wave simulations. See [14] for an implementation of the original Serre equations with bathymetry. Furthermore, we present a new splitting method for the gSerre equations, based on Riemann invariants. This method is applicable to any equation resulting from a small perturbation of a hyperbolic system of two equations. From it we write an effective fully-nonlinear one way system: although we are describing one-way propagation, the dynamics cannot be written as a single first order equation since strongly nonlinear wavespeeds would require nonlocal corrections in a first order equation. The use of Riemann invariants also allows one to choose initial conditions in the Serre equations that result in one-way propagation.

### Acknowledgements

This work has been partially supported by ANR HEXECO, Project n° BLAN07 – 1\_192661, and by the 2008 Framework Program for Research, Technological development and Innovation of the Cyprus Research Promotion Foundation under the Project AΣTI/0308(BE)/05. The second author wishes to thank the Centre National de la Recherche Scientifique (CNRS) for sponsoring his visit to Ecole Normale Supérieure de Cachan in 2008-2009, and support from the National Science Foundation under Grant DMS-0604635.

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### Figure Captions

Figure 1: Speed-amplitude curves for the solitary waves of the gSerre equations with no surface tension. For comparison, the KdV and Euler results are shown.

Figure 2: Wave profiles (top) and velocity (bottom) for Serre and gSerre solitary waves of equal amplitude. For gSerre, the profile shown is the limiting profile for  $\alpha^2 = 0.15$  (see Figure 1).

Figure 3: Wave profiles (top) and velocity (bottom) for capillary-gravity solitary waves. For the case  $B = 1/2$  there are four curves since two different amplitude solutions are shown.

Figure 4: Time evolution of (3) and (4) with  $\alpha^2 = 2$  and  $B = 0$ . Upper curves are  $h$  and lower curves are  $u$ . Thicker lines denote the initial condition and the solution at the final time.

Figure 5: Time evolution of (3) and (4) with  $\alpha^2 = 2$  and  $B = 1/2$ . Upper curves are  $h$  and lower curves are  $u$ . Thicker lines denote the initial condition and the solution at the final time.

Figure 6: Time evolution of right-traveling waves in the one-way Serre equation (6) with (7) where initially  $L = 0$  and  $R$  is a Gaussian centered at  $x = 25$ . The evolution shows the initial breakup into solitary waves. The dotted line is  $L$ , which remains small at later times and travels to the *right*.

Figure 7: Time evolution of (1) and (2) with  $\alpha^2 = 2$  and  $B = 0$  demonstrating the effectiveness of using  $R^- = 0$  in the initialization of right-traveling waves. Upper curves are  $h$  and lower curves are  $u$ . Thicker lines denote the initial condition and the solution at the final time.

## Figures

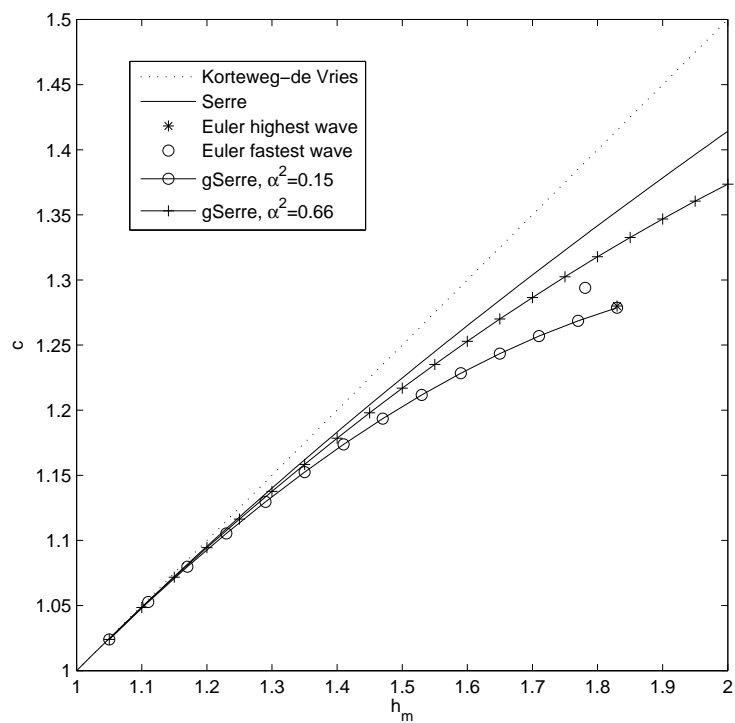


FIG. 1:

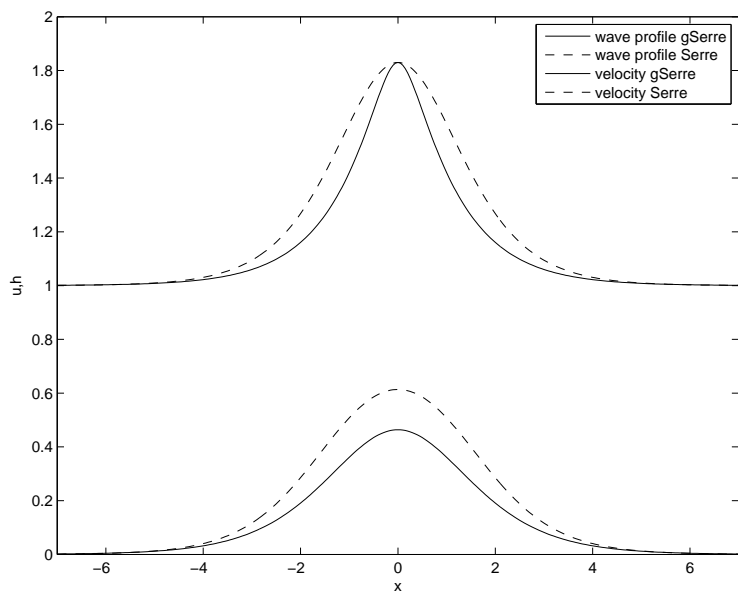


FIG. 2:

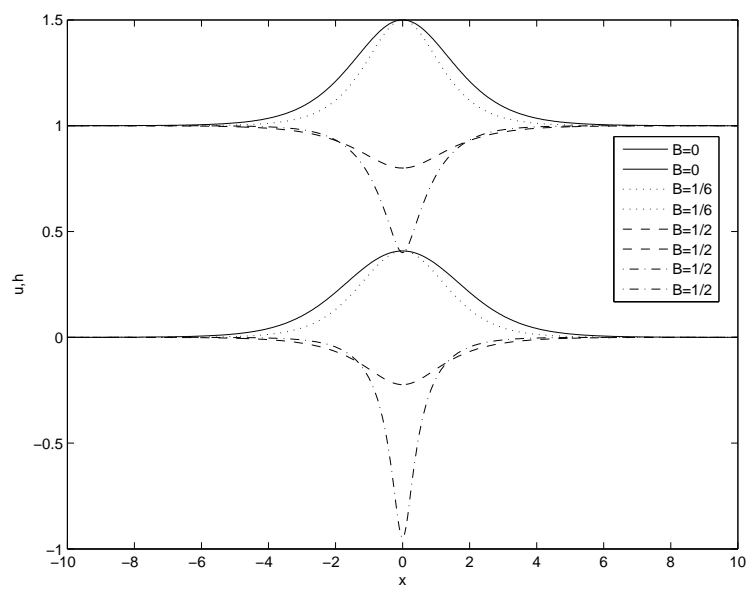


FIG. 3:

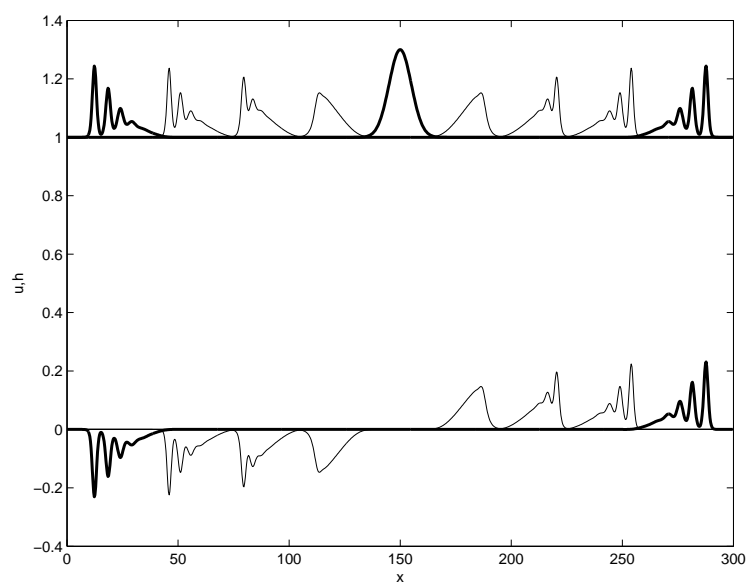


FIG. 4:

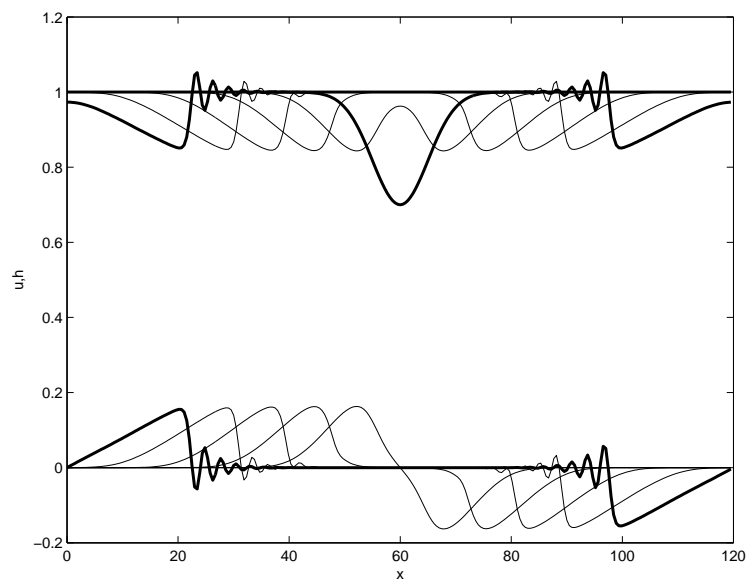


FIG. 5:

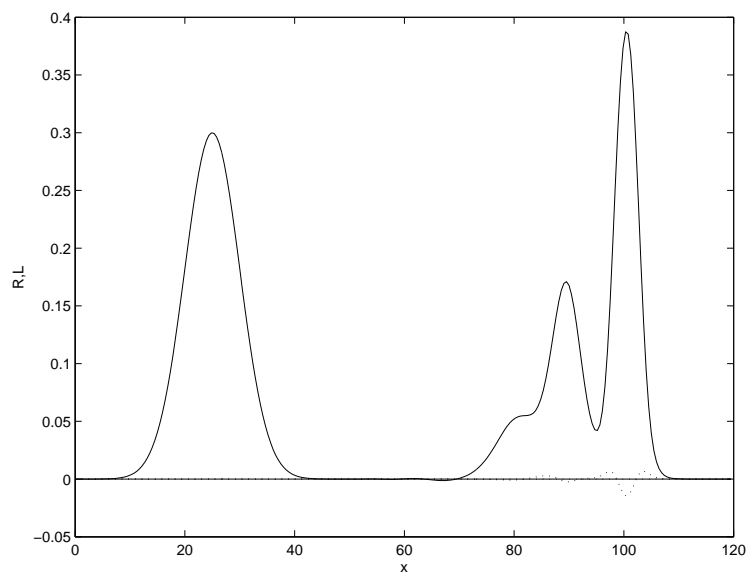


FIG. 6:

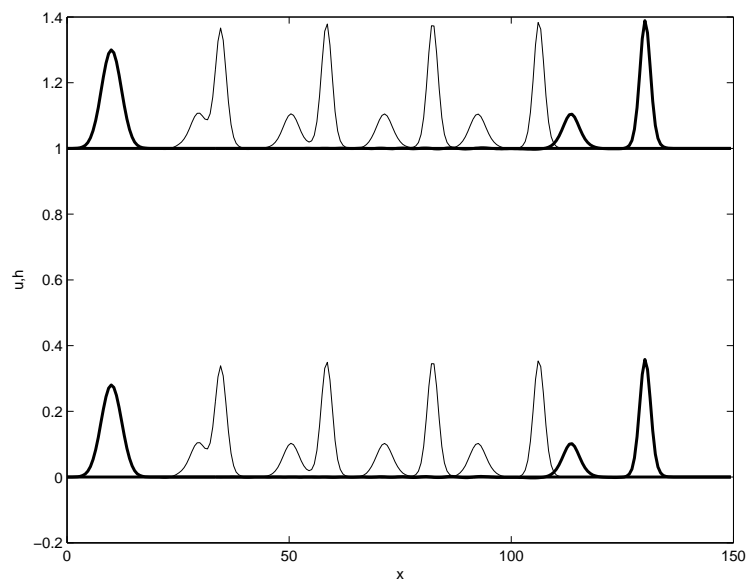


FIG. 7: