2. Review of the Lyapunov inverse iteration. In this section we review the algorithm for detecting Hopf points proposed in [18] and the mathematical theory on which the algorithm is built. The following theorem is the main theoretical motivation for the techniques in [18].

THEOREM 2.1. *Assume* $\mathbf{M}$ *is nonsingular, and assume* μ_1, μ_2 $(\mu_1 \neq \mu_2)$ *are simple eigenvalues of* (1.2) *whose corresponding eigenvectors are* x_1, x_2. *The following two statements are equivalent:*

1. *zero is a double eigenvalue of* $\mathcal{J} \otimes \mathbf{M} + \mathbf{M} \otimes \mathcal{J}$ *that corresponds to the eigenvector* $\xi_1 x_1 \otimes x_2 + \xi_2 x_2 \otimes x_1$ *for any* $\xi_1, \xi_2 \in \mathbb{C}$;
2. (μ_1, μ_2) *is the only pair of eigenvalues of* (1.2) *that sums to zero.*

Proof. Since $\mathbf{M}$ is nonsingular, by properties of Kronecker products, $\mathbf{M} \otimes \mathbf{M}$ is nonsingular. Also using properties of Kronecker products, we can show that

$$(\mathbf{M} \otimes \mathbf{M})^{-1}(\mathcal{J} \otimes \mathbf{M} + \mathbf{M} \otimes \mathcal{J})(x_i \otimes x_j) = (\mu_i + \mu_j)(x_i \otimes x_j). \tag{2.1}$$

Thus, the eigenpairs of $(\mathbf{M} \otimes \mathbf{M})^{-1}(\mathcal{J} \otimes \mathbf{M} + \mathbf{M} \otimes \mathcal{J})$ are $(\mu_i + \mu_j, \xi_i x_i \otimes x_j + \xi_j x_j \otimes x_i)$ $(i, j = 1, 2, \ldots, n)$ for any $\xi_i, \xi_j \in \mathbb{C}$.

We first prove statement 2 given statement 1. Note that $\mathcal{J} \otimes \mathbf{M} + \mathbf{M} \otimes \mathcal{J}$ and $(\mathbf{M} \otimes \mathbf{M})^{-1}(\mathcal{J} \otimes \mathbf{M} + \mathbf{M} \otimes \mathcal{J})$ have the same null space. Thus, if statement 1 is true, zero is also a double eigenvalue of $(\mathbf{M} \otimes \mathbf{M})^{-1}(\mathcal{J} \otimes \mathbf{M} + \mathbf{M} \otimes \mathcal{J})$ with the eigenvector $\xi_1 x_1 \otimes x_2 + \xi_2 x_2 \otimes x_1$. By (2.1), (μ_1, μ_2) is the only pair of eigenvalues of (1.2) that sums to zero.

Now assume statement 2 is true. Since (μ_1, μ_2) $(\mu_1 \neq \mu_2)$ is the only pair of eigenvalues of (1.2) that sums to zero and both μ_1, μ_2 are simple, by (2.1), zero is a double eigenvalue of $(\mathbf{M} \otimes \mathbf{M})^{-1}(\mathcal{J} \otimes \mathbf{M} + \mathbf{M} \otimes \mathcal{J})$ with the eigenvector $\xi_1 x_1 \otimes x_2 + \xi_2 x_2 \otimes x_1$. Since $\mathcal{J} \otimes \mathbf{M} + \mathbf{M} \otimes \mathcal{J}$ and $(\mathbf{M} \otimes \mathbf{M})^{-1}(\mathcal{J} \otimes \mathbf{M} + \mathbf{M} \otimes \mathcal{J})$ have the same null space, statement 1 follows immediately. ∎

We continue to assume that $\mathbf{M}$ is nonsingular. In addition, assume that when $\lambda = \lambda_c$, the rightmost eigenvalues of (1.3) (βi and $-\beta i$) are simple and there are no other eigenvalues lying on the imaginary axis. Let v and $\overline{v}$ be the eigenvectors corresponding to βi and $-\beta i$. Since λ_c is the parameter closest to zero such that (1.3) has a pair of eigenvalues that sums to zero, according to Theorem 2.1, λ_c is the parameter closest to zero such that $(\mathbf{A}+\lambda\mathbf{B})\otimes\mathbf{M}+\mathbf{M}\otimes(\mathbf{A}+\lambda\mathbf{B})$ has a zero eigenvalue. Alternatively, λ_c is the eigenvalue closest to zero for the $n^2 \times n^2$ generalized eigenvalue problem

$$(\Delta_1 + \lambda \Delta_0) z = 0, \tag{2.2}$$

where

$$\Delta_1 = \mathbf{A} \otimes \mathbf{M} + \mathbf{M} \otimes \mathbf{A},$$
$$\Delta_0 = \mathbf{B} \otimes \mathbf{M} + \mathbf{M} \otimes \mathbf{B}.$$

Note that by Theorem 2.1, the eigenvector corresponding to λ_c is $z_c = \xi_1 v \otimes \overline{v} + \xi_2 \overline{v} \otimes v$. Therefore, finding λ_c, the quantity that allows us to estimate the critical parameter value α_c, is equivalent to finding the eigenvalue of (2.2) with smallest modulus. One standard approach for computing this eigenvalue is to use an iterative method such as inverse iteration for (2.2). This approach is obviously impractical for large-scale problems since inverse iteration requires solution of linear systems with coefficient matrix Δ_1, which has order n^2. We can use properties of Kronecker products to rewrite (2.2) into a linear equation of $n \times n$ matrices. In particular, let $Z \in \mathbb{R}^{n \times n}$

be such that $z = vec(Z)$ (see [16, p. 244]). Then it is known (see [16, p. 255]) that (2.2) is equivalent to (1.4). Therefore, finding λ with smallest modulus for (2.2) is equivalent to finding λ with smallest modulus for (1.4). Because of the relationship between (2.2) and (1.4), we will refer to λ as an eigenvalue and Z as an eigenvector of (1.4). The following theorem from [18] describes the properties of Z.

THEOREM 2.2. *Assume that* λ *is a real eigenvalue of* (2.2). *If* (1.3) *has eigenpairs* $(\beta i, v)$ *and* $(-\beta i, \overline{v})$ $(\beta{>}0)$ *and no other eigenvalues on the imaginary axis, then* (1.4) *has a real symmetric eigenvector of rank two, namely,* $Z = vv^* + \overline{v}v^T$, *which is unique up to a scalar factor and is semidefinite, and a unique skew-symmetric eigenvector of rank two, namely,* $Z = vv^* - \overline{v}v^T$.

It is suggested in [18] that we should restrict our computation to the real symmetric eigenspace of (1.4). Under this restriction, the eigenvalue of interest, λ_c, is simple. The corresponding eigenvector, which is symmetric and of rank two, has a natural representation in the form of a truncated eigenvalue decomposition $Z_c = \mathcal{V}\mathcal{D}\mathcal{V}^T$, where $\mathcal{V} \in \mathbb{R}^{n\times 2}$ is orthonormal and $\mathcal{D} \in \mathbb{R}^{2\times 2}$ is diagonal. By Theorem 2.2, $\text{span}\{\mathcal{V}\} = \text{span}\{v, \overline{v}\}$. Therefore, once we find λ_c and its eigenvector Z_c for (1.4), the rightmost eigenvalues of (1.3) can be found easily by solving the 2×2 problem

$$\mathcal{V}^T(\mathbf{A} + \lambda_c\mathbf{B})\mathcal{V}y = \mu\mathcal{V}^T\mathbf{M}\mathcal{V}y. \tag{2.3}$$

The associated eigenvectors are $v = \mathcal{V}y$, $\overline{v} = \mathcal{V}\overline{y}$. To find the eigenvalue closest to zero for (1.4), a version of inverse iteration can be applied.

ALGORITHM 1 (inverse iteration for (1.4)).

1. Given $V_1 \in \mathbb{R}^n$ with $\|V_1\|_2 = 1$ and $D_1 = 1$, let $Z_1 = V_1D_1V_1^T$.
2. For $j = 1, 2, \ldots$
   2.1. Compute the eigenvalue approximation[1]

$$\lambda_j = -\frac{\text{trace}(\widetilde{A}_j^T D_j \widetilde{M}_j D_j + \widetilde{M}_j^T D_j \widetilde{A}_j D_j)}{\text{trace}(\widetilde{B}_j^T D_j \widetilde{M}_j D_j + \widetilde{M}_j^T D_j \widetilde{B}_j D_j)}, \tag{2.4}$$

   where

$$\widetilde{A}_j = V_j^T\mathbf{A}V_j, \quad \widetilde{B}_j = V_j^T\mathbf{B}V_j, \quad \widetilde{M}_j = V_j^T\mathbf{M}V_j. \tag{2.5}$$

   2.2. If (λ_j, Z_j) is accurate enough, then stop.
   2.3. Else, solve

$$\mathbf{A}Y_j\mathbf{M}^T + \mathbf{M}Y_j\mathbf{A}^T = F_j \tag{2.6}$$

   in factored form $Y_j = V_{j+1}D_{j+1}V_{j+1}^T$, where $F_j = \mathbf{B}Z_j\mathbf{M}^T + \mathbf{M}Z_j\mathbf{B}^T$.
   2.4. Normalize: $D_{j+1} \leftarrow D_{j+1}/\|D_{j+1}\|_F$. Let $Z_{j+1} = V_{j+1}D_{j+1}V_{j+1}^T$.

If $\mathbf{A}$ is nonsingular, then (2.6) is equivalent to the *Lyapunov equation*

$$SY_j + Y_jS^T = \mathbf{A}^{-1}F_j\mathbf{A}^{-T}, \tag{2.7}$$

where $S = \mathbf{A}^{-1}\mathbf{M}$. Let $\text{rank}(Z_j) = k$; it is reasonable to assume that $k \ll n$ (see [18]). The right-hand side of (2.7) can be represented by its truncated eigenvalue decomposition

$$\mathbf{A}^{-1}F_j\mathbf{A}^{-T} = P_jC_jP_j^T, \tag{2.8}$$

[1]The Rayleigh quotient (2.4) can be derived using a property of Kronecker products (see [16, Exercise 25, p. 252]).