

Branch Cuts in MAPLE 17

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Abstract

Accurate and comprehensible knowledge about the position of branch cuts is essential for correctly working with multi-valued functions, such as the square root and logarithm. We discuss the new tools in MAPLE 17 for calculating and visualising the branch cuts of such functions, and others built up from them. The cuts are described in an intuitive and accurate form, offering substantial improvement on the descriptions previously available.

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1 Introduction

When defining multi-valued functions (such as the natural logarithm and the square root) choices must be made for the positioning of the branch cuts. These usually follow [1], or its modern counterpart [10]. MAPLE agrees for all elementary functions except arccot, for reasons explained in [5]. There are different, often unstated, viewpoints for dealing with multi-valued functions [6]. MAPLE along with most computer algebra software (and indeed most users) works with multi-valued functions by defining their single-valued counterparts which will have discontinuities over the branch cuts. It is important that users of these functions understand the position of such cuts.

For this purpose we built the `BranchCuts` package, now integrated into MAPLE 17 as part of the `FunctionAdvisor` project [4]. The code calculates and visualises the branch cuts of functions of a single complex variable. It is used by default in MAPLE 17 while readers with older versions can download a version with the basic functionality from <http://opus.bath.ac.uk/32511/>.

In [9] we presented in detail the generic algorithms which can be used for calculating branch cuts. Here we focus on the practical improvements offered by the MAPLE implementation.

2 Branch cuts in MAPLE

Information on MAPLE's predefined functions may be accessed with the `FunctionAdvisor` tool: a handbook for special functions, designed to be both human and machine readable, and to process output to fit the query [4]. We can view the defining branch cuts for a given function by querying the name without argument. For example,

```
> FunctionAdvisor(branch_cuts, ln);
```

$$[\ln(z), z < 0] \tag{1}$$

indicates the logarithm has a branch cut along the negative real axis.

In MAPLE 16 executing the command on a function with argument would lead to the same output with z replaced by this argument (modulo automatic simplification). For example,

```
> FunctionAdvisor(branch_cuts, ln(z^2));
```

$$[\ln(z^2), z^2 < 0] \tag{2}$$

indicating that the function has a branch cut when z^2 takes negative real values. From this we can infer that the cuts are along the imaginary axis excluding zero. The statement is accurate but it would be more useful to return the imaginary axis directly. Further, consider the example,

```
> FunctionAdvisor(branch_cuts, ln(-sqrt(z)));
```

$$[\ln(-\sqrt{z}), -\sqrt{z} < 0] \tag{3}$$

from which we infer that the function has a branch cut along the positive real axis. However, the function is also discontinuous along the negative real axis due to the extra branch cut introduced by the square root. Figure 1 presents 3d plots of the imaginary part of these three functions making the discontinuities clear. We display the imaginary part as crossing the branch cut causes the logarithm to change by $2\pi i$. For other functions such as arctan the discontinuity occurs in the real part. Pairs of 3d plots for the real and imaginary parts are now available directly from the `FunctionAdvisor` using the extra argument `plot=3d`. The position of branch cuts may be inferred from such plots, but MAPLE 17 is able to give accurate and intuitive algebraic descriptions and hence 2d visualisations.

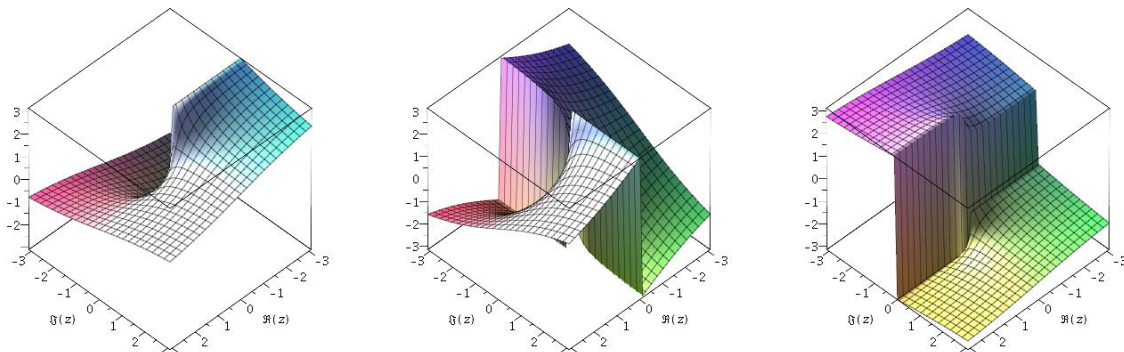


Figure 1: Plots of the imaginary parts of the three functions discussed in Section 2.

3 Improved representing and calculation of branch cuts

Two approaches have been implemented for calculating branch cuts. The first moves the problem from one complex variable to two real variables: $z = \Re(z) + i\Im(z)$. The resulting semi-algebraic system is then solved, usually using cylindrical algebraic decomposition (CAD), so that each branch cut is represented by an equation setting one of the variables in terms of the other and inequations bounding the independent variable. Hence in MAPLE 17 the second example from Section 2 becomes

```
> FunctionAdvisor(branch_cuts, ln(z^2));
```

$$[\ln(z^2), \Re(z) = 0 \text{ And } 0 < \Im(z), \Re(z) = 0 \text{ And } \Im(z) < 0] \quad (2')$$

which is as accurate as (2) but is more useful for the user and subsequent computations.

When the argument is not polynomial a second approach is preferred where cuts are stored as a (possibly complex and transcendental) function of a real parameter with a given range, as in [8]. This is achieved by setting the argument to the parameter and inverting to find solutions for the variable using MAPLE's `solve` capabilities, applying with a range determined by the defining cut. We can compute the branch cuts of algebraic expressions using all the mathematical functions of the language for which the branch cuts are known. For situations like sums and products both approaches take the union of the individual cuts. Hence in MAPLE 17 the following output is obtained for the third example, which is both more intuitive and more accurate than (3).

```
> FunctionAdvisor(branch_cuts, ln(-sqrt(z)));
```

$$[\ln(-\sqrt{z}), z = \alpha^2 \text{ And } \alpha \in \text{RealRange}(-\infty, 0), z < 0] \quad (3')$$

More details on the algorithms are available in [9]. In MAPLE 17 2d visualisations of the algebraic descriptions of the branch cuts may be obtained directly from the `FunctionAdvisor` with the optional argument `plot=2d`. An example is on the left of Figure 2.

The code considers any univariate function with defining cuts known to the `FunctionAdvisor` including elementary functions and their inverses. It also covers multivariate functions with branch cuts in one variable only, and hence those which come with parameters such as the Bessel functions and Chebyshev polynomials. We note that understanding the position of branch cuts can also have applications in computer algebra itself, for example in the safe application of identities. Consider

$$2 \arcsin(z) = \arcsin(2z\sqrt{1-z^2}). \quad (4)$$

A visualisation of the branch cuts of (4) produced from the new algebraic description is given on the left of Figure 2 with the other images plots of the imaginary part of LHS(4)–RHS(4). Identity (4) is true within the hour glass shaped region bounded by the cuts and false otherwise. Algorithmic approaches to identifying such regions of truth using branch cuts and CAD have been studied in a series of papers starting with [2]. Recent progress was reported in [7] and we note that the new CAD algorithm in [3] is very well suited for dealing with input from the first branch cut representation.

4 When is a branch cut not a branch cut?

Failure could occur due to time/memory constraints, or the `solve` tools being unable to identify all solutions. However, for all practical examples encountered these problems do not occur. A more prevalent issue is branch cuts being returned which do not correspond to discontinuities of the input. We call these *spurious cuts* and they have two causes. First, when functions are combined the discontinuities introduced may cancel. Second, when the argument of a function contains fractional powers then the solve tools may find solutions corresponding to all possible roots rather than the specific root concerned. In MAPLE 17 a cautious approach is used, including branch cuts that may be spurious rather than risk losing true cuts. We note that spurious cuts may be identified visually for example by comparing the 2d visualisation of the algebraic output with 3d plots oriented to appear 2d (as on the right of Figure 2). This orientation can be obtained interactively with the mouse or directly from the `FunctionAdvisor` using the argument `plot=32d`. Methods to systematically identify and remove these spurious cuts from the algebraic output are currently under development.

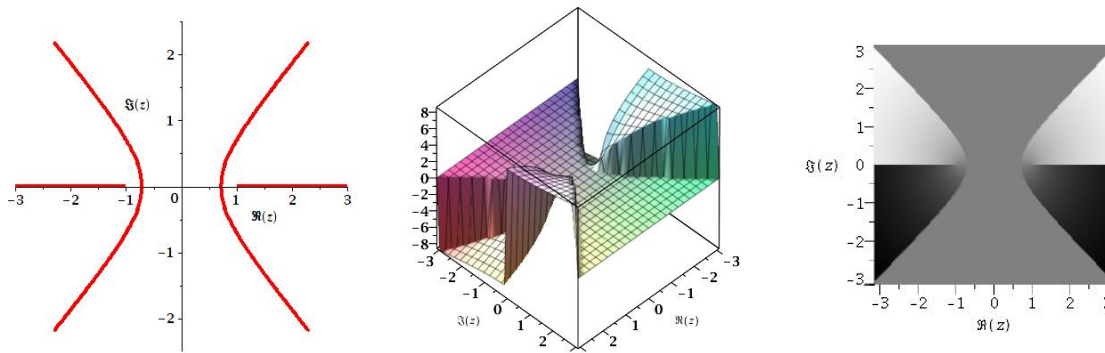


Figure 2: Visualising the branch cuts of equation (4).

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