

SPECTRAL ANALYSIS OF ONE-DIMENSIONAL HIGH-CONTRAST ELLIPTIC PROBLEMS WITH PERIODIC COEFFICIENTS*

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Abstract. We study the behavior of the spectrum of a family of one-dimensional operators with periodic high-contrast coefficients as the period goes to zero, which may represent, e.g., the elastic or electromagnetic response of a two-component composite medium. Compared to the standard operators with moderate contrast, they exhibit a number of new effects due to the underlying nonuniform ellipticity of the family. The effective behavior of such media in the vanishing period limit also differs notably from that of multidimensional models investigated thus far by other authors, due to the fact that neither component of the composite forms a connected set. We then discuss a modified problem, where the equation coefficient is set to a positive constant on an interval that is independent of the period. Formal asymptotic analysis and numerical tests with finite elements suggest the existence of localized eigenfunctions (“defect modes”), whose eigenvalues are situated in the gaps of the limit spectrum for the unperturbed problem.

Key words. elliptic differential equations, homogenization, spectrum

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1. Introduction.

1.1. The general context for the problem in hand. The description of the effective behavior of high-contrast composites (“high-contrast homogenization”) has been of particular interest in the analysis and applied mathematics communities over the last decade. The analytical part of the related literature starts with the work [18], which developed in detail some earlier ideas of [1] concerning the use of “two-scale convergence” for the analysis of the limit behavior of the boundary-value problem

$$-\operatorname{div}(\mathcal{A}^\varepsilon(x/\varepsilon)\nabla u) = f, \quad f \in L^2(\Omega), \quad u \in H_0^1(\Omega), \quad \mathcal{A}^\varepsilon = \varepsilon^2\chi_0 I + \chi_1 I, \quad \varepsilon > 0,$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, and χ_0, χ_1 are the indicator functions of $[0, 1)^n$ -periodic sets in \mathbb{R}^n such that $\chi_0 + \chi_1 = 1$.

Several contributions to the high-contrast homogenization followed—in the linear and nonlinear scalar and vector contexts, with various sets of assumptions about the underlying geometry of the composite. With applications mainly in solid mechanics and electromagnetism, high-contrast media have served as a theoretical ground for a number of effects observed in physics experiments, in particular those related to

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photonic band-gap materials and cloaking metamaterials [15]. The range of techniques developed in these contexts and their applications continue their rapid expansion, and the present paper is aimed at addressing some aspects that have thus far been left out of the scope of the related research.

More specifically, we approach the question of the analysis of the spectral behavior of high-contrast composites when the component represented by the function χ_1 (the “matrix” of the composite) is disconnected in \mathbb{R}^n . Clearly, this is always so in one dimension ($n = 1$). In the present article we study this particular case.

1.2. Problem set-up. We consider solutions u to the following family of elliptic problems on an interval $(a, b) \subset \mathbb{R}$:

$$(1.1) \quad A^\varepsilon u - \lambda u = f, \quad f \in L^2(a, b), \quad \varepsilon > 0, \quad \lambda \in \mathbb{C},$$

where the operators A^ε are given by the closed sesquilinear form

$$(1.2) \quad (A^\varepsilon u, v) = \int_a^b p(x/\varepsilon)(\varepsilon^2 \chi_0(x/\varepsilon) + \chi_1(x/\varepsilon)) u'(x) \overline{v'(x)} dx, \quad u, v \in \mathfrak{H}.$$

Here $p = p(y) > 0$ is a 1-periodic function in \mathbb{R} such that $p, p^{-1} \in L^\infty(0, 1)$, the functions χ_0 and χ_1 are the indicator functions of 1-periodic open sets F_0 and F_1 such that $\overline{F_0} \cup \overline{F_1} = \mathbb{R}$, and \mathfrak{H} denotes a closed linear subspace of $H^1(a, b)$ that contains $C_0^\infty(a, b)$. We make no assumptions regarding boundedness of the interval (a, b) ; in particular, it may coincide with the whole space \mathbb{R} .

In applied contexts the problem (1.1) corresponds to, e.g., the study of wave propagation in a layered two-dimensional or three-dimensional composite structure where $f = 0$, $\lambda > 0$. In what follows we study the spectrum S^ε of the problem (1.1), i.e., the set of values of λ for which $A^\varepsilon - \lambda I$ does not have a bounded inverse in $L^2(a, b)$. Throughout the article we employ the notation $\sigma(A)$ for the spectrum of an operator A and the notation Q for the “unit cell” $[0, 1)$ whenever we describe the behavior with respect to the “physical” variables x, y . We continue writing $[0, 1)$ for the “Floquet–Bloch dual” cell when we refer to the domain of the quasimomentum θ .

1.3. Our strategy for the analysis of (1.1). It has been well understood in the existing literature on the subject (see [2], [18], [20]) that in the analysis of convergence of spectra of families of differential operators with periodic rapidly oscillating coefficients one has to deal with two distinct issues: the lower semicontinuity of the spectra in the sense of Hausdorff convergence of sets and the possibility of spectral pollution, the lack of which is often referred to as “spectral completeness.” The former issue, which in the wider spectral analysis context has been looked at from a more general perspective (see, e.g., [4]), is usually dealt with by first proving a variant of the strong resolvent convergence. In the case of periodic operators involving multiple scales, one typically makes use of two-scale convergence (see, e.g., [14], [1], [18]). In the present paper we follow this general approach in proving the related lower semicontinuity statements both for the whole-space problem and for the problem in a bounded interval. This part of the analysis of spectral convergence does not eliminate the need for a study of spectral completeness: unless some assumptions are made concerning the geometry of the periodic composite in question (see, e.g., [18]), one may not get the best possible “lower bound” for the limit spectrum. It has been noticed that in order to capture the behavior with respect to all Bloch components in the limit as $\varepsilon \rightarrow 0$, it is preferable to use an advanced, “multicell,” version of the standard two-scale convergence; see, e.g., [2], [6, Chapter 5], where this more refined

approach is adopted. It is a version of this last, more detailed, procedure that we develop in the present article.

In the proof of spectral completeness, a natural strategy seems to be to analyze the relative strength of different Bloch components in a given (convergent) sequence of eigenfunctions. This idea has been elaborated in [2] in the specific context of “high-frequency” homogenization, with the use of what the authors refer to as the “Bloch measures.” A combination of a compactness argument in the related space of measures and a special “slow-variable modulation” construction then yields the simultaneous convergence of the given sequence to a limit eigenfunction and of the associated eigenvalues. In the present work we deal with a situation where such a compactness argument does not suffice, since the limit of an eigenfunction sequence may have a nontrivial part in the orthogonal complement (in the two-scale version of the L^2 space) to functions that are constant in the matrix component. (In the set-up studied in [2] this orthogonal complement is zero.) Once we have suitable control of this part, we can prove spectral convergence for some operator families not amenable to the approach of [2], including those considered in [17], [18], and [6, Chapter 4].

The key element in our analysis, which allows us to implement the above idea, is Lemma 3.2 below (see section 3.1). This statement establishes a uniform version of the Poincaré-type inequality between the projection of a given function onto the “poorly behaving” subspace of H^1 and the L^2 -norm of its derivative on the part of the domain where solutions of the eigenvalue problem can be shown to be a priori small as $\varepsilon \rightarrow 0$. Different versions of the same idea have appeared in a number of other contexts, serving a similar purpose of somehow compensating for the apparent loss of compactness in the problem, for example, in the form of the Korn inequality in elasticity (see, e.g., [7] and also [19] for its multiscale versions), in the form of the energy method in classical homogenization (see [13]), and, more recently, in the form of a generalized Weyl decomposition for problems with degeneracies (see [11]). For nonlinear variants of the same idea, the reader is referred to the geometric rigidity (see [8]) and \mathcal{A} -quasiconvexity (see [9]).

For an easier introduction to the problem, in what follows we start with the analysis of the problem (1.1) in the whole-space case, $(a, b) = \mathbb{R}$; see section 2. While a version of the compactness argument developed in the bounded-interval setting (see section 3) applies here as well (once complemented by a suitable Weyl-sequence argument), we present a different argument, based on some ideas of [6, Chapter 5], where the spectral analysis is carried out in a more challenging setting of the Maxwell system.

Throughout this paper we assume for simplicity that the restriction of χ_0 to the periodicity cell $[0, 1)$ is the indicator function of an open interval (α, β) , which we also denote by Q_0 . We use the notation Q_1 for the interior of the complement of Q_0 to the interval $(0, 1)$.

2. Limit analysis for the whole space: $(a, b) = \mathbb{R}$. Let us consider for a moment the case $p \equiv 1$. One well-known procedure for calculating S^ε , the spectrum of the problem (1.1), is the Floquet–Bloch decomposition (see, e.g., [3]) following the rescaling $y = x/\varepsilon$. For $\theta \in [0, 1)$, let $\lambda = \lambda(\theta)$ be the eigenvalues corresponding to θ -quasiperiodic eigenfunctions of the differential expression $((\chi_0 + \varepsilon^{-2}\chi_1)u)'$ on the interval $(0, 1)$. Such eigenvalues are obtained by solving the dispersion equation

$$\frac{1}{2} \left(\frac{1}{\varepsilon} + \varepsilon \right) \sin \left(\varepsilon \sqrt{\lambda} (\alpha - \beta + 1) \right) \sin \left(\sqrt{\lambda} (\alpha - \beta) \right)$$

$$+ \cos(\varepsilon\sqrt{\lambda}(\alpha - \beta + 1)) \cos(\sqrt{\lambda}(\alpha - \beta)) = \cos(2\pi\theta).$$

Passing to the limit in the above equation as $\varepsilon \rightarrow 0$ yields

$$(2.1) \quad \frac{1}{2}(\alpha - \beta + 1)\sqrt{\lambda} \sin(\sqrt{\lambda}(\alpha - \beta)) + \cos(\sqrt{\lambda}(\alpha - \beta)) = \cos(2\pi\theta).$$

By varying θ as indicated, we obtain (for $\alpha = 1/4$, $\beta = 3/4$) the set shown in Figure 1.

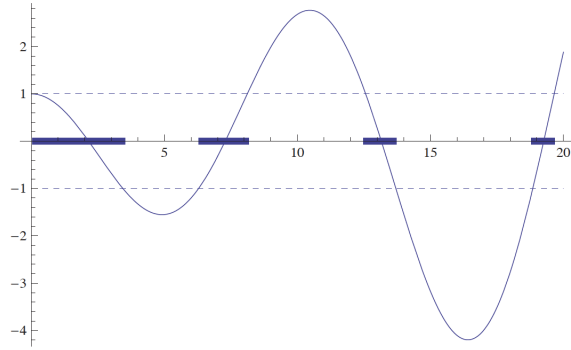


FIG. 1. The square root of the limit Bloch spectrum for $p \equiv 1$. The oscillating solid line is the graph of the function $f(t) = \cos(t/2) - t \sin(t/2)/4$, where t represents $\sqrt{\lambda}$ in the formula (2.1) with $\alpha = 1/4$, $\beta = 3/4$. The square root of the spectrum is the union of the intervals, indicated by bold lines, that correspond to $t \in \mathbb{R}^+$ such that $-1 \leq f(t) \leq 1$ (the so-called pass bands).

The following statement is a particular case of our main result, Theorem 2.2 below.

PROPOSITION 2.1. *Let $(a, b) = \mathbb{R}$, $p \equiv 1$. Then the set $\lim_{\varepsilon \rightarrow 0} S^\varepsilon$ is given by the union of solution sets for (2.1) for all $\theta \in [0, 1)$.*

In what follows we consider the case of an arbitrary p while adopting the above Floquet–Bloch approach to the description of the set $\lim_{\varepsilon \rightarrow 0} S^\varepsilon$ of limit points of S^ε as $\varepsilon \rightarrow 0$. For $\theta \in [0, 1)$, we denote by $H_\theta^1(Q)$ the space of functions $u \in H^1(Q)$ that are θ -quasiperiodic, i.e., such that $v(y) = \exp(2\pi i \theta y) u(y)$, $y \in Q$, for some¹ $u \in H_\#^1(Q)$. We also denote

$$(2.2) \quad V(\theta) := \{v \in H_\theta^1(Q) : v'(y) = 0 \text{ for } y \in Q_1\}.$$

Note that $V(\theta)$ is a closed subspace of $H^1(Q)$ and is therefore a Hilbert space when inheriting the standard H^1 -norm. The sesquilinear form $\mathcal{A}_\theta : V(\theta) \times V(\theta) \rightarrow \mathbb{C}$ defined by

$$\mathcal{A}_\theta(u, v) := \int_{Q_0} p u' \overline{v'}$$

¹As the notation $H_0^1(Q)$ is usually reserved for the space of $H^1(Q)$ functions vanishing on the boundary of Q , we denote by $H_\#^1(Q)$ the space $H_\theta^1(Q)$ when $\theta = 0$.

is clearly closed and nonnegative on $V(\theta)$. Therefore, it defines a self-adjoint operator $A(\theta)$ (see, e.g., [12]) such that

$$(2.3) \quad (A(\theta)u, v) = \mathcal{A}_\theta(u, v) \quad \forall v \in V(\theta), \quad u \in \text{dom}(A(\theta)) \subset V(\theta),$$

where the domain of $A(\theta)$, denoted by $\text{dom}(A(\theta))$, is a dense subset of $V(\theta)$ with respect to the L^2 -norm. Henceforth (\cdot, \cdot) denotes the usual inner product in $L^2(Q)$. Under the adopted conditions on the coefficient p , the operator $A(\theta)$ is self-adjoint and has a compact inverse (except for the case $\theta = 0$, when it has a compact inverse as an operator on $V(\theta) \ominus \mathbb{C}$). Therefore, the spectrum $\sigma(A(\theta))$ is discrete and unbounded; i.e., it consists of eigenvalues $0 \leq \lambda_1(\theta) \leq \lambda_2(\theta) \leq \dots$ of finite multiplicity with eigenfunctions $v^k(\theta) = v^k(\theta, y)$. The eigenfunctions corresponding to different eigenvalues are automatically orthogonal in $L^2(Q)$. We also carry out the orthogonalization process on those eigenfunctions that correspond to the same eigenvalue, and we normalize each eigenfunction so that $\|v^k(\theta)\|_{L^2(Q)} = 1$ for all $\theta \in [0, 1)$, $k \in \mathbb{N}$.

Our aim within this section is to show that the set $\lim_{\varepsilon \rightarrow 0} S^\varepsilon$ coincides with the union of the spectra of the operators $A(\theta)$, $\theta \in [0, 1)$, i.e.,

$$(2.4) \quad \lim_{\varepsilon \rightarrow 0} S^\varepsilon = \bigcup_{\theta \in [0, 1)} \sigma(A(\theta)).$$

More precisely, we establish the following theorem.

THEOREM 2.2.

1. For a given $\theta \in [0, 1)$ let $\lambda \in \sigma(A(\theta))$. There exist $\lambda_\varepsilon \in S^\varepsilon$ such that $\lambda_\varepsilon \rightarrow \lambda$ as $\varepsilon \rightarrow 0$.
2. Let $\lambda_\varepsilon \in S^\varepsilon$ be such that $\lambda_\varepsilon \rightarrow \lambda \in \mathbb{R}$. Then there exist $\theta \in [0, 1)$ and $u \in V(\theta) \setminus \{0\}$ such that

$$(2.5) \quad \int_\alpha^\beta pu'\overline{\varphi'} = \lambda \int_0^1 u\overline{\varphi} \quad \forall \varphi \in V(\theta).$$

In order to demonstrate property 1 we use an appropriate modification of the strong two-scale resolvent convergence, introduced in [18]. By showing, for each $N \in \mathbb{N}$, that a subsequence of A^ε strongly two-scale resolvent converges to an “intermediate” operator A_N on the space of L^2_{loc} -functions that are NQ -periodic, we establish property 1 for $\theta = j/N$, $0 \leq j \leq N-1$. The details of this argument, which rely on a procedure that we refer to as “ NQ -periodic homogenization,” are given in Appendix A.

An essential ingredient in extending property 1 to hold for $\theta \in [0, 1)$ and in proving Property 2 is the following continuity property of the set $V(\theta)$ with respect to θ .

LEMMA 2.3. *The family $V(\theta)$ is continuous in $\theta \in [0, 1)$ in the following sense. For fixed $\theta \in [0, 1)$, let $\theta_\varepsilon \in [0, 1)$ be such that $\theta_\varepsilon \rightarrow \theta$ as $\varepsilon \rightarrow 0$. Then for any $\varphi \in V(\theta)$ there exist $\varphi_\varepsilon \in V(\theta_\varepsilon)$ such that $\varphi_\varepsilon \rightarrow \varphi$ strongly in $H^1(Q)$ as $\varepsilon \rightarrow 0$.*

Proof. For $\theta \in [0, 1)$ the space $V(\theta)$ consists of functions that are θ -quasiperiodic and constant in each connected component of Q_1 ; that is, for any $\varphi \in V(\theta)$ one has $\varphi(y) = \eta(\theta, y)c + v(y)$, $y \in Q$, where $c \in \mathbb{C}$, $v \in H^1_0(\alpha, \beta)$, and

$$\eta(\theta, y) := \begin{cases} 1, & y \in [0, \alpha), \\ (\exp(2\pi i\theta) - 1)(\beta - \alpha)^{-1}(y - \alpha) + 1, & y \in [\alpha, \beta], \\ \exp(2\pi i\theta), & y \in (\beta, 1). \end{cases}$$

For each value of ε we now define φ_ε by the formula $\varphi_\varepsilon(y) = \eta(\theta_\varepsilon, y)c + v(y)$, $y \in Q$. Notice that by construction $\varphi_\varepsilon \in V(\theta_\varepsilon)$, and, since η is uniformly continuous with respect to θ , one has $\varphi_\varepsilon \rightarrow \varphi$ strongly in $H^1(Q)$. \square

Next we prove Theorem 2.2.

Proof. Property 1: We shall outline the proof here and refer the reader to Appendix A for the full details. Note that property 1 holds for rational θ , the set $\lim_{\varepsilon \rightarrow 0} S^\varepsilon$ is closed, and the rationals are dense in $[0, 1]$. Therefore, it suffices to show that the eigenvalues $\lambda(\theta)$ of $A(\theta)$ are continuous with respect to θ . Indeed, in Appendix B we show this to follow from Lemma 2.3, i.e., from the continuity of $V(\theta)$.

Property 2: For each $\varepsilon > 0$, since $\lambda_\varepsilon \in S^\varepsilon$, by the Floquet–Bloch decomposition there exists $u_\varepsilon \in H_{\theta_\varepsilon}^1(\varepsilon Q) \setminus \{0\}$ such that

$$(2.6) \quad \int_{\varepsilon Q} p(x/\varepsilon)(\varepsilon^2 \chi_0(x/\varepsilon) + \chi_1(x/\varepsilon))u'_\varepsilon(x)\overline{\varphi'(x)}dx = \lambda_\varepsilon \int_{\varepsilon Q} u_\varepsilon(x)\overline{\varphi(x)}dx$$

for all $\varphi \in H_{\theta_\varepsilon}^1(\varepsilon Q)$. Rescaling the formulation (2.6) with $y = x/\varepsilon$ yields the existence of $v_\varepsilon \in H_{\theta_\varepsilon}^1(Q)$, $\|v_\varepsilon\|_{L^2(Q)} = 1$, such that

$$(2.7) \quad \varepsilon^{-2} \int_{Q_1} p(y)v'_\varepsilon(y)\overline{\varphi'(y)}dy + \int_{Q_0} p(y)v'_\varepsilon(y)\overline{\varphi'(y)}dy = \lambda_\varepsilon \int_Q v_\varepsilon(y)\overline{\varphi(y)}dy$$

for all $\varphi \in H_{\theta_\varepsilon}^1(Q)$.

The sequence θ_ε is bounded, and therefore there exists some $\theta \in [0, 1]$ such that, up to a subsequence which we do not relabel, $\theta_\varepsilon \rightarrow \theta$. Without loss of generality, if $\theta = 1$ we set $\theta = 0$ so that $\theta \in [0, 1]$. By substituting $\varphi = v_\varepsilon$ into (2.7), the sequence v_ε satisfies the bounds

$$(2.8) \quad \|\chi_1 v'_\varepsilon\|_{L^2(Q)} \leq C\varepsilon, \quad \|\chi_0 v'_\varepsilon\|_{L^2(Q)} \leq C,$$

with a constant $C > 0$ independent of ε .

Due to the weak compactness of bounded sets in $H^1(Q)$, the bounds (2.8), along with $\|v_\varepsilon\|_{L^2(Q)} = 1$, imply that, up to extracting a subsequence, v_ε converges weakly in $H^1(Q)$, and therefore strongly in $L^2(Q)$, to some $v_0 \in H^1(Q)$, $\|v_0\|_{L^2(Q)} = 1$. Clearly, for $w_\varepsilon(y) := \exp(-2\pi i \theta_\varepsilon y)v_\varepsilon(y)$, $y \in Q$, one has $w_\varepsilon \in H_{\theta_\varepsilon}^1(Q)$, and the uniform convergence of $\exp(2\pi i \theta_\varepsilon \cdot)$ to $\exp(2\pi i \theta \cdot)$ as $\varepsilon \rightarrow 0$ implies that w_ε converges weakly in $H_{\theta_\varepsilon}^1(Q)$ to w_0 given by the formula $w_0(y) = \exp(-2\pi i \theta y)v_0(y)$, $y \in Q$, such that $v_0 \in H_\theta^1(Q)$. Furthermore, (2.8) implies that $\chi_1 v'_\varepsilon \rightarrow 0$ strongly in $L^2(Q)$; hence $v_0 \in V(\theta)$.

In order to show that v_0 satisfies the limit identity (2.5), for a fixed $\varphi_0 \in V(\theta)$ let $\varphi_\varepsilon \in V(\theta_\varepsilon)$ be given by Lemma 2.3. Substituting φ_ε into (2.7), we obtain

$$(2.9) \quad \int_\alpha^\beta p v'_\varepsilon \overline{\varphi'_\varepsilon} = \lambda_\varepsilon \int_0^1 v_\varepsilon \overline{\varphi_\varepsilon}.$$

By virtue of the facts that $\varphi_\varepsilon \rightarrow \varphi_0$ strongly in $H^1(Q)$ and $v_\varepsilon \rightharpoonup v_0$ weakly in $H^1(Q)$, passing to the limit $\varepsilon \rightarrow 0$ in (2.9) immediately implies (2.5) for $u = v_0$. \square

Remark 2.1. The above “limit spectrum” $\lim_{\varepsilon \rightarrow 0} S^\varepsilon$ is strictly larger than the set obtained by the two-scale analysis of the operator A^ε of the paper [18]. In particular, the spectrum of the one-dimensional version of the homogenized operator obtained in [18] coincides with $\{\lambda_k(0)\}_{k=1}^\infty$, using our notation.

Our analysis above shows that the set $\lim_{\varepsilon \rightarrow 0} S^\varepsilon$ has, in fact, a band-gap structure, with infinitely many gaps opening in the interval $[0, \infty)$, as $\varepsilon \rightarrow 0$. This fact suggests possible applications of the above composite structures to the design of optical or acoustic band-gap materials, which we discuss in section 5. The above effect also raises a mathematical question regarding the analysis of the limit behavior of the operators A^ε in the case when (a, b) is a bounded interval, which we study in the next section.

3. Spectral behavior on a bounded interval. It is known that the classical, “moderate-contrast,” analogue of the problem (1.1)–(1.2) leads to limit spectra of different kinds for problems on bounded and unbounded intervals (a, b) : the limit set in the case of the problem in the whole space is purely absolutely continuous, while in the case $-\infty < a < b < \infty$ it is purely discrete; i.e., it consists of eigenvalues with finite multiplicities; see, e.g., [3]. A similar situation occurs in multidimensional high-contrast problems where the inclusion $F_0 \cap Q$ has a nonzero distance to the boundary of Q ; see [18], where, in addition, some eigenvalues of infinite multiplicity are present. As we shall see below, this is not the case for the problem (1.1)–(1.2), when the spectrum of the operator A^ε defined by (1.2) contains the right-hand side of (2.4), i.e., the spectrum of the operator defined by the form (1.2) with (a, b) replaced by \mathbb{R} and \mathfrak{H} replaced by $H^1(\mathbb{R})$. This makes the limit spectrum in question considerably richer than that described in [18].

In this section we employ, for convenience, the following notation: $\Omega := (a, b)$, $\Omega_1^\varepsilon := \Omega \cap (\varepsilon F_1)$, $\Omega_0^\varepsilon := \Omega \cap (\varepsilon F_0)$, $\tilde{\Omega}^\varepsilon := \varepsilon^{-1}\Omega$, $\tilde{\Omega}_1^\varepsilon := \varepsilon^{-1}\Omega_1^\varepsilon$, and $\tilde{\Omega}_0^\varepsilon := \varepsilon^{-1}\Omega_0^\varepsilon$.

3.1. The main convergence result. The following theorem holds.

THEOREM 3.1. *Consider an operator A^ε from the class described in section 1.2. The limit set $\lim_{\varepsilon \rightarrow 0} S^\varepsilon$ is given by the union $\sigma_{\text{Bloch}} \cup \sigma_{\text{boundary}}$, where*

$$\sigma_{\text{Bloch}} := \bigcup_{\theta \in [0, 1)} \sigma(A(\theta))$$

(cf. (2.4)), and

$$\sigma_{\text{boundary}} := \left\{ \lambda \in \mathbb{R} : \lambda = \lim_{\varepsilon \rightarrow 0} \lambda_\varepsilon, A^\varepsilon u_\varepsilon = \lambda_\varepsilon u_\varepsilon, u_\varepsilon \in \mathfrak{H}, \|u_\varepsilon\|_{L^2(a, b)} = 1, \right.$$

$$\left. \|u_\varepsilon\|_{L^2(a+\delta_\varepsilon, b-\delta_\varepsilon)} \rightarrow 0 \text{ for some } \delta_\varepsilon \rightarrow 0 \text{ as } \varepsilon \rightarrow 0 \right\}.$$

An essential element to proving Theorem 3.1 is validating the following statement.

LEMMA 3.2. *There exists a constant $C > 0$, which depends on α and β only, such that for all $\theta \in [0, 1)$ one has*

$$(3.2) \quad \|w\|_{H^1(Q)} \leq C \|w'\|_{L^2(Q_1)} \quad \forall w \in V^\perp(\theta).$$

Here $V^\perp(\theta)$ denotes the orthogonal complement of $V(\theta)$ (see (2.2)) in the space $H_\theta^1(Q)$.

Proof. It suffices to show that the statement of the lemma holds for the equivalent H^1 -norm

$$\| |u| \| := \left(\left| \int_{Q_1} u \right|^2 + \|u'\|_{L^2(Q)}^2 \right)^{1/2}.$$

Indeed, assuming that the statement of the lemma holds for the norm $||| \cdot |||$, with a constant $C = \tilde{C}$ in the analogue of (3.2) for $||| \cdot |||$, and $\hat{c} := \sup_{|||w|||=1} \|w\|_{H^1(Q)}$, then for all $w \in H_\theta^1(Q)$ we obtain

$$\inf_{v \in V(\theta)} \|w - v\|_{H^1(Q)} \leq \hat{c} \inf_{v \in V(\theta)} |||w - v||| \leq \hat{c}\tilde{C} \|w'\|_{L^2(Q_1)},$$

e.g., the claim of the lemma with $C = \hat{c}\tilde{C}$. In what follows we keep the notation $V^\perp(\theta)$ for the orthogonal complement to $V(\theta)$ with respect to the inner product induced by $||| \cdot |||$. We first note some properties of functions that belong to the space $V^\perp(\theta)$, which follow immediately from the characterization of the space $V(\theta)$ given in the proof of Lemma 2.3.

PROPOSITION 3.3. *Let $w \in V^\perp(\theta)$; then the following hold.*

(i) *The equation $w''(y) = 0$ holds for $y \in Q_0$. In particular, the function w is linear on the Q_0 -component of the unit cell:*

$$w(y) = (w(\beta) - w(\alpha))(\beta - \alpha)^{-1}(y - \alpha) + w(\alpha), \quad y \in Q_0.$$

(ii) *The formula*

$$\int_{Q_1} w = \frac{(w(\beta) - w(\alpha))(1 - \exp(-2\pi i\theta))}{(\beta - \alpha)(\alpha + (1 - \beta)\exp(-2\pi i\theta))}$$

holds.

We now return to the proof of Lemma 3.2. We consider three different cases, depending on the location of the quasimomentum θ within the Floquet–Bloch cell $[0, 1)$.

Case I: $\theta = 0$. Using Proposition 3.3, we find that

$$|||w|||^2 = \int_{Q_1} |w'|^2 + \frac{|w(\beta) - w(\alpha)|^2}{\beta - \alpha}.$$

Since $w(1) = w(0)$, we obtain the estimate

$$\begin{aligned} |||w|||^2 &\leq \|w'\|_{L^2(Q_1)}^2 + \frac{2}{\beta - \alpha} (|w(\beta) - w(1)|^2 + |w(0) - w(\alpha)|^2) \\ &= \|w'\|_{L^2(Q_1)}^2 + \frac{2}{\beta - \alpha} \left(\left| \int_\beta^1 w' \right|^2 + \left| \int_0^\alpha w' \right|^2 \right) \\ &\leq \left(1 + \frac{2|Q_1|}{\beta - \alpha} \right) \|w'\|_{L^2(Q_1)}^2. \end{aligned}$$

Case II: $\theta \in (0, \delta) \cup (1 - \delta, 1)$, where $0 < \delta < 1/2$ is to be chosen appropriately. Again, Proposition 3.3 implies that

$$\begin{aligned} |||w|||^2 &= \left| \int_{Q_1} w \right|^2 + \|w'\|_{L^2(Q_1)}^2 + \frac{|w(\beta) - w(\alpha)|^2}{\beta - \alpha} \\ &= \left(\frac{1}{\beta - \alpha} + \frac{1}{|d_\theta|^2} \right) |w(\beta) - w(\alpha)|^2 + \|w'\|_{L^2(Q_1)}^2, \end{aligned}$$

where $d_\theta := (\beta - \alpha)(1 - \exp(-2\pi i\theta))^{-1}(\alpha + (1 - \beta)\exp(-2\pi i\theta))$. From the fact that $w(1) = \exp(2\pi i\theta)w(0)$ we infer

$$\begin{aligned} |w(\beta) - w(\alpha)|^2 &\leq 3 \left(|w(1) - w(\beta)|^2 + |w(\alpha) - w(0)|^2 + |w(1) - w(0)|^2 \right) \\ &\leq 3 |Q_1| \|w'\|_{L^2(Q_1)}^2 + 3 |\exp(2\pi i\theta) - 1|^2 |w(0)|^2, \end{aligned}$$

and therefore

$$\begin{aligned} \|w\|^2 &\leq \left(1 + 3|Q_1| \left(\frac{1}{\beta - \alpha} + \frac{1}{|d_\theta|^2} \right) \right) \|w'\|_{L^2(Q_1)}^2 \\ &\quad + 3 \left(\frac{1}{\beta - \alpha} + \frac{1}{|d_\theta|^2} \right) |\exp(2\pi i\theta) - 1|^2 |w(0)|^2. \end{aligned}$$

Notice that $|d_\theta|^2 = (\beta - \alpha)^2(2 - 2\cos(2\pi\theta))^{-1}(\alpha^2 + (1 - \beta)^2 + 2\alpha(1 - \beta)\cos(2\pi\theta))$; hence $|d_\theta|$ vanishes at $\theta = 1/2$ for the special case $\alpha = 1 - \beta$. In view of this observation and in order to have a bound below on $|d_\theta|$, we require that $\delta < 1/4$. Further, by continuity of the embedding of $H^1(Q)$ in $C(\bar{Q})$, there exists a constant \hat{c} , which is independent of θ , such that

$$|w(0)| \leq \hat{c} \|w\|,$$

and thus

$$\begin{aligned} (3.3) \quad \|w\|^2 &\leq \left(1 + 3|Q_1| \left(\frac{1}{\beta - \alpha} + \frac{1}{|d_\theta|^2} \right) \right) \|w'\|_{L^2(Q_1)}^2 \\ &\quad + 3 \left(\frac{1}{\beta - \alpha} + \frac{1}{|d_\theta|^2} \right) |\exp(2\pi i\theta) - 1|^2 \hat{c}^2 \|w\|^2. \end{aligned}$$

We now choose $\delta < 1/4$ so that $3 \left((\beta - \alpha)^{-1} + |d_\theta|^{-2} \right) |\exp(2\pi i\theta) - 1|^2 \hat{c}^2 < 1/2$, and hence $|d_\theta|^{-2}$ is bounded above by a constant independent of θ in the intervals considered. The inequality (3.3) now immediately implies the required estimate.

Case III: $\theta \in [\delta, 1 - \delta]$. For given $x \in (\beta, 1], y \in [0, \alpha)$ the identities

$$w(x) = \int_\beta^x w' + w(\beta), \quad w(y) = - \int_y^\alpha w' + w(\alpha)$$

imply, in view of Proposition 3.3(ii),

$$w(x) - w(y) = w(\beta) - w(\alpha) + \left(\int_\beta^x + \int_y^\alpha \right) w' = d_\theta \int_{Q_1} w + \left(\int_\beta^x + \int_y^\alpha \right) w'.$$

In particular, substituting $x = 1, y = 0$ and using the fact that $w(1) = \exp(2\pi i\theta)w(0)$, we obtain

$$w(1) = \frac{d_\theta}{1 - \exp(-2\pi i\theta)} \int_{Q_1} w + \frac{1}{1 - \exp(-2\pi i\theta)} \int_{Q_1} w',$$

whence

$$w(x) = - \int_x^1 w' + w(1) = - \int_x^1 w' + \frac{1}{1 - \exp(-2\pi i\theta)} \left(d_\theta \int_{Q_1} w + \int_{Q_1} w' \right).$$

Integrating the last identity over $(\beta, 1]$ yields

$$(3.4) \quad \int_{\beta}^1 w = - \int_{\beta}^1 \left(\int_x^1 w' \right) dx + \frac{1 - \beta}{1 - \exp(-2\pi i \theta)} \left(d_{\theta} \int_{Q_1} w + \int_{Q_1} w' \right).$$

Similarly, we write

$$w(y) = \int_0^y w' + \frac{\exp(-2\pi i \theta)}{1 - \exp(-2\pi i \theta)} \left(d_{\theta} \int_{Q_1} w + \int_{Q_1} w' \right),$$

which upon integration over $(0, \alpha)$ yields

$$(3.5) \quad \int_0^{\alpha} w = \int_0^{\alpha} \left(\int_0^y w' \right) dy + \frac{\alpha \exp(-2\pi i \theta)}{1 - \exp(-2\pi i \theta)} \left(d_{\theta} \int_{Q_1} w + \int_{Q_1} w' \right).$$

Combining (3.4) and (3.5), we obtain

$$\begin{aligned} \left(1 - \frac{(1 - \beta + \alpha \exp(-2\pi i \theta)) d_{\theta}}{1 - \exp(-2\pi i \theta)} \right) \int_{Q_1} w &= \int_0^{\alpha} \left(\int_0^y w' \right) dy \\ &\quad - \int_{\beta}^1 \left(\int_x^1 w' \right) dx + \frac{(1 - \beta + \alpha \exp(-2\pi i \theta))}{1 - \exp(-2\pi i \theta)} \int_{Q_1} w'. \end{aligned}$$

Squaring both sides and using the Cauchy–Schwarz inequality yields

$$\begin{aligned} \left| 1 - \frac{(1 - \beta + \alpha \exp(-2\pi i \theta)) d_{\theta}}{1 - \exp(-2\pi i \theta)} \right|^2 \left| \int_{Q_1} w \right|^2 \\ \leq 2 \left(4 + \frac{|1 - \beta + \alpha \exp(-2\pi i \theta)|^2}{|1 - \exp(-2\pi i \theta)|^2} \right) \|w'\|_{L^2(Q_1)}^2. \end{aligned}$$

A direct calculation shows that the coefficient on the left-hand side of the last inequality is separated from zero in the range of θ considered. Similarly, the coefficient on the right-hand side is bounded.

Finally, we argue that

$$\begin{aligned} |w(\beta) - w(\alpha)| &\leq |w(\beta) - w(1)| + |w(0) - w(\alpha)| + |w(0)| + |w(1)| \\ &= \left| \int_{\beta}^1 w' \right| + \left| \int_0^{\alpha} w' \right| + |w(0)| + |w(1)|, \end{aligned}$$

and

$$(1 - \beta)w(1) = \int_{\beta}^1 \left(\int_x^1 w' \right) dx + \int_{\beta}^1 w, \quad \alpha w(0) = - \int_0^{\alpha} \left(\int_0^x w' \right) dx + \int_0^{\alpha} w.$$

The required inequality follows since by Proposition 3.3(i) one has

$$\|w\|^2 = \left| \int_{Q_1} w \right|^2 + \frac{|w(\beta) - w(\alpha)|^2}{\beta - \alpha} + \|w'\|_{L^2(Q_1)}^2.$$

This completes the proof of Lemma 3.2. \square

We now prove Theorem 3.1.

Proof. The inclusion $\bigcup_{\theta \in [0,1]} \sigma(A(\theta)) \subset \lim_{\varepsilon \rightarrow 0} S^\varepsilon$ is proved in the same way as in the case of the whole-space problem; see the proof of Theorem 2.2. In what follows we therefore discuss the converse inclusion; i.e., for $\lambda_\varepsilon \in S^\varepsilon$, $\lambda_\varepsilon \rightarrow \lambda$ such that $\lambda \notin \sigma_{\text{boundary}}$ one has $\lambda \in \bigcup_{\theta \in [0,1]} \sigma(A(\theta))$. Let us consider such a sequence λ_ε . Notice first that for each $\lambda_\varepsilon \in S^\varepsilon$, there exist $u_\varepsilon \in \mathfrak{H}$, $\|u_\varepsilon\|_{L^2(a,b)} = 1$, such that for any sequence $\delta_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} 0$, there exists a constant $c > 0$ uniformly bounding the L^2 -norms of u_ε away from the boundary of Ω ,

$$\|u_\varepsilon\|_{L^2(a+\delta_\varepsilon, b-\delta_\varepsilon)} \geq c \quad \forall \varepsilon,$$

and the identity

$$\int_{\Omega_\varepsilon^\delta} p\left(\frac{x}{\varepsilon}\right) u'_\varepsilon(x) \overline{\varphi'(x)} dx + \varepsilon^2 \int_{\Omega_\varepsilon^\delta} p\left(\frac{x}{\varepsilon}\right) u'_\varepsilon(x) \overline{\varphi'(x)} dx = \lambda_\varepsilon \int_{\Omega} u_\varepsilon(x) \overline{\varphi(x)} dx$$

holds for all $\varphi \in \mathfrak{H}$. Indeed, if this were not the case, then λ would be an element of σ_{boundary} ; see (3.1). Rescaling the above statement with $y = x/\varepsilon$ yields the existence of functions $v_\varepsilon \in H^1(\tilde{\Omega}^\varepsilon)$, $\|v_\varepsilon\|_{L^2(\tilde{\Omega}^\varepsilon)} = 1$, such that

$$(3.6) \quad \|v_\varepsilon\|_{L^2(\varepsilon^{-1}a+\varepsilon^{-1}\delta_\varepsilon, \varepsilon^{-1}b-\varepsilon^{-1}\delta_\varepsilon)} \geq c \quad \forall \varepsilon,$$

and

$$(3.7) \quad \varepsilon^{-2} \int_{\tilde{\Omega}_\varepsilon^\delta} p(y) v'_\varepsilon(y) \overline{\varphi'(y)} dy + \int_{\tilde{\Omega}_\varepsilon^\delta} p(y) v'_\varepsilon(y) \overline{\varphi'(y)} dy = \lambda_\varepsilon \int_{\tilde{\Omega}^\varepsilon} v_\varepsilon(y) \overline{\varphi(y)} dy$$

for all $\varphi \in \tilde{\mathfrak{H}} := \{v \mid v(y) = u(\varepsilon y) \text{ for some } u \in \mathfrak{H}\}$. Choosing $\varphi = v_\varepsilon$ in (3.7) yields the estimates (a priori bounds)

$$(3.8) \quad \|v'_\varepsilon\|_{L^2(\tilde{\Omega}_\varepsilon^\delta)} \leq \varepsilon C_B, \quad \|v'_\varepsilon\|_{L^2(\tilde{\Omega}_\varepsilon^\delta)} \leq C_B,$$

where $C_B > 0$ is independent of ε .

Step 1. Without loss of generality, we assume that $\overline{\tilde{\Omega}^\varepsilon} \subset N_\varepsilon Q$ for some $N_\varepsilon \in \mathbb{N}$, such that $\varepsilon N_\varepsilon \rightarrow 1$ as $\varepsilon \rightarrow 0$. We also take a sequence $\delta_\varepsilon > 0$ such that $\varepsilon^{-1}\delta_\varepsilon = q_\varepsilon \in \mathbb{N}$, $\delta_\varepsilon \rightarrow 0$, and $q_\varepsilon^{-1} = \varepsilon\delta_\varepsilon^{-1} \rightarrow 0$ as $\varepsilon \rightarrow 0$, and we denote by χ_ε a smooth ‘‘cut-off’’ function such that $0 \leq \chi_\varepsilon \leq 1$, $\chi_\varepsilon \in C_0^\infty(\tilde{\Omega}^\varepsilon)$, with the additional properties that $\chi_\varepsilon = 1$ on $(\varepsilon^{-1}a + q_\varepsilon, \varepsilon^{-1}b - q_\varepsilon)$ and $\chi'_\varepsilon = 0$ on $F_1 \cap N_\varepsilon Q$. Note that such a function χ_ε varies from 0 to 1 on the interval $(\varepsilon^{-1}a, \varepsilon^{-1}a + q_\varepsilon)$, which contains q_ε integer translations of the interval (α, β) , and similarly varies from 1 to 0 on the interval $(\varepsilon^{-1}b - q_\varepsilon, \varepsilon^{-1}b)$. Therefore, it can be constructed by piecing together translations of smooth monotone functions g on $[0, 1]$ that are constant on the intervals $[0, \alpha)$, $(\beta, 1]$ and satisfy the inequality $|g'(x)| \leq 2(\beta - \alpha)^{-1}q_\varepsilon^{-1}$ for $x \in [\alpha, \beta]$. Clearly in this case the bound

$$(3.9) \quad \|\chi'_\varepsilon\|_{L^\infty(\tilde{\Omega}^\varepsilon)} \leq 2(\beta - \alpha)^{-1}q_\varepsilon^{-1}$$

holds. In particular, $\|\chi'_\varepsilon\|_{L^\infty(N_\varepsilon Q)} \rightarrow 0$ as $\varepsilon \rightarrow 0$, where we extend the function χ_ε by zero outside the interval $\tilde{\Omega}^\varepsilon$.

For every $N \in \mathbb{N}$ we introduce the function space (cf. (2.2))

$$V_N := \{u \in H_{\#}^1(NQ) : u'(y) = 0 \text{ for } y \in F_1\}$$

and denote V_N^\perp to be the orthogonal complement of V_N in the space $H_{\#}^1(NQ)$ equipped with the usual inner product. For each ε we set the $\chi_\varepsilon v_\varepsilon$ to take zero values on $N_\varepsilon Q \setminus \tilde{\Omega}^\varepsilon$, and we consider the functions $U_\varepsilon := P_{V_{N_\varepsilon}}(\chi_\varepsilon v_\varepsilon)$, $V_\varepsilon := P_{V_{N_\varepsilon}^\perp}(\chi_\varepsilon v_\varepsilon)$, where $P_{V_{N_\varepsilon}}$ and $P_{V_{N_\varepsilon}^\perp}$ denote the orthogonal projections in the space $H_{\#}^1(N_\varepsilon Q)$ onto its subspaces V_{N_ε} and $V_{N_\varepsilon}^\perp$, respectively. Note that, due to the normalization of v_ε and (3.6), the bounds

$$(3.10) \quad 1 \geq \|\chi_\varepsilon v_\varepsilon\|_{L^2(\tilde{\Omega}^\varepsilon)} \geq c$$

hold for all ε . Further, we show that there exists a constant C_\perp , independent of ε , such that

$$(3.11) \quad \|V_\varepsilon\|_{H^1(N_\varepsilon Q)} \leq (c_0 q_\varepsilon^{-1} + \varepsilon C_B) C_\perp.$$

The inequality (3.11) is a consequence of (3.9) and the following proposition.

PROPOSITION 3.4. *There exists a constant $C_\perp > 0$ such that*

$$\|P_{V_N^\perp} u\|_{H^1(NQ)} \leq C_\perp \|u'\|_{L^2(F_1 \cap NQ)}$$

for any $N \in \mathbb{N}$ and any $u \in H_{\#}^1(NQ)$.

Indeed, if Proposition 3.4 holds, then (3.11) follows from the a priori bounds (3.8). Note that Lemma 3.2 is one of the key ingredients in the proof of Proposition 3.4, which we give next.

Proof. Consider a function $u \in V_N^\perp$ for some positive integer N . Notice that for each $j = 0, 1, \dots, N-1$ the function

$$u_j(x) = N^{-1} \sum_{k=0}^{N-1} u(x+k) \exp(-2\pi i j k / N)$$

belongs to the space $V(\theta_j)^\perp$, $\theta_j := j/N$. Indeed, for any $v \in V(\theta_j) \subset H_{\theta_j}^1(Q)$ it is clear that v belongs to the space V_N when extended in a quasiperiodic fashion. Further, for any $j = 0, 1, \dots, N-1$, one has

$$\begin{aligned} \int_Q u_j \bar{v} &= N^{-1} \sum_{k=0}^{N-1} \int_0^1 u(x+k) \exp(-2\pi i j k / N) \overline{v(x)} dx \\ &= N^{-1} \sum_{k=0}^{N-1} \int_k^{k+1} u(x) \exp(-2\pi i j k / N) \overline{v(x-k)} dx = N^{-1} \int_0^N u \bar{v}, \end{aligned}$$

and similarly

$$\int_Q u_j' \bar{v}' = N^{-1} \int_0^N u' \bar{v}'.$$

Combining the above two identities yields

$$\int_Q u_j \bar{v} + \int_Q u_j' \bar{v}' = 0 \quad \forall v \in V(\theta_j),$$

as required.

Now using the Parseval identity and Lemma 3.2 we obtain

$$\|u\|_{H^1(NQ)}^2 = N \sum_{j=0}^{N-1} \|u_j\|_{H^1(Q)}^2 \leq CN \sum_{j=0}^{N-1} \|u'_j\|_{L^2(Q_1)}^2 = C \|u'\|_{L^2(F_1 \cap NQ)}^2,$$

where the positive constant C does not depend on N . \square

Step 2. There exists a constant \widehat{C} independent of ε such that

$$(3.12) \quad \|U'_\varepsilon\|_{L^2(N_\varepsilon Q)} \leq \widehat{C}.$$

Indeed, by the a priori bounds (3.8), as well as (3.9) and (3.11), we find that

$$\begin{aligned} \|U'_\varepsilon\|_{L^2(N_\varepsilon Q)} &\leq \|V'_\varepsilon\|_{L^2(N_\varepsilon Q)} + \|\chi'_\varepsilon v_\varepsilon + \chi_\varepsilon v'_\varepsilon\|_{L^2(N_\varepsilon Q)} \\ &\leq (\varepsilon C_B + c_0 q_\varepsilon^{-1}) C_\perp + c_0 q_\varepsilon^{-1} + (1 + \varepsilon) C_B. \end{aligned}$$

Further, the identity $\chi_\varepsilon v_\varepsilon = U_\varepsilon + V_\varepsilon$ and the bounds (3.10) imply

$$(3.13) \quad c - (\varepsilon C_B + c_0 q_\varepsilon^{-1}) C_\perp \leq \|U_\varepsilon\|_{L^2(N_\varepsilon Q)} \leq 1 + (\varepsilon C_B + c_0 q_\varepsilon^{-1}) C_\perp.$$

As the functions U_ε belong to the spaces V_{N_ε} , by the discrete Floquet–Bloch transform the following decomposition holds: $U_\varepsilon = \sum_{j=0}^{N_\varepsilon-1} U_\varepsilon^j$ for $U_\varepsilon^j \in V(j/N_\varepsilon)$ given by

$$U_\varepsilon^j(x) = N_\varepsilon^{-1} \sum_{k=0}^{N_\varepsilon-1} U_\varepsilon(x+k) \exp(-2\pi i j k / N_\varepsilon), \quad x \in N_\varepsilon Q, \quad j = 0, 1, \dots, N_\varepsilon - 1.$$

Recalling, from section 2, that for fixed ε the eigenfunctions $v^k(j/N_\varepsilon)$ form a complete sequence in the $L^2(Q)$ -closure of the set $V(j/N_\varepsilon)$, we can decompose U_ε^j with respect to this sequence, e.g.,

$$(3.14) \quad \begin{aligned} U_\varepsilon(x) &= \sum_{j=0}^{N_\varepsilon-1} \sum_{k=1}^{\infty} \widehat{U}_\varepsilon^k\left(\frac{j}{N_\varepsilon}\right) v^k\left(\frac{j}{N_\varepsilon}, x\right), \\ U'_\varepsilon(x) &= \sum_{j=0}^{N_\varepsilon-1} \sum_{k=1}^{\infty} \widehat{U}_\varepsilon^k\left(\frac{j}{N_\varepsilon}\right) (v^k)'\left(\frac{j}{N_\varepsilon}, x\right), \end{aligned}$$

where $\widehat{U}_\varepsilon^k(j/N_\varepsilon) \in \mathbb{C}$. The Parseval identity

$$N_\varepsilon \sum_{j=0}^{N_\varepsilon-1} \sum_{k=1}^{\infty} \left| \widehat{U}_\varepsilon^k\left(\frac{j}{N_\varepsilon}\right) \right|^2 = \|U_\varepsilon\|_{L^2(N_\varepsilon Q)}^2$$

and (3.13) imply

$$(3.15) \quad (c - (\varepsilon C_B + c_0 q_\varepsilon^{-1}) C_\perp)^2 \leq N_\varepsilon \sum_{j=0}^{N_\varepsilon-1} \sum_{k=1}^{\infty} \left| \widehat{U}_\varepsilon^k\left(\frac{j}{N_\varepsilon}\right) \right|^2 \leq (1 + (\varepsilon C_B + c_0 q_\varepsilon^{-1}) C_\perp)^2.$$

Denoting by $\delta(\cdot - \theta)$ the Dirac mass at θ , the inequalities (3.15) can be rewritten as

$$(3.16) \quad (c - (\varepsilon C_B + c_0 q_\varepsilon^{-1}) C_\perp)^2 \leq \sum_{k=1}^{\infty} \int_0^1 d\mu_\varepsilon^k \leq (1 + (\varepsilon C_B + c_0 q_\varepsilon^{-1}) C_\perp)^2,$$

$$\text{where } d\mu_\varepsilon^k(\theta) := N_\varepsilon \sum_{j=0}^{N_\varepsilon-1} \left| \hat{U}_\varepsilon^k\left(\frac{j}{N_\varepsilon}\right) \right|^2 \delta\left(\theta - \frac{j}{N_\varepsilon}\right) d\theta.$$

Clearly, for each k the sequence $\{\mu_\varepsilon^k\}_\varepsilon$ is bounded in the space of Radon measures on $[0, 1]$. Therefore, up to a subsequence, μ_ε^k weakly converges as $\varepsilon \rightarrow 0$ to some measure μ^k , e.g.,

$$(3.17) \quad \int_0^1 u d\mu_\varepsilon^k \rightarrow \int_0^1 u d\mu^k \quad \forall u \in C[0, 1].$$

The above result follows from recalling that the space of finite Radon measures on $[0, 1]$ coincides with the dual space $C[0, 1]^*$, and hence bounded sets of Radon measures are relatively compact with respect to weak star convergence in this space.

Step 3. Here we show that

$$(3.18) \quad c \leq \sum_{k=1}^{\infty} \int_0^1 d\mu^k \leq 1,$$

which implies in particular that there exists at least one nonzero measure μ^{k_0} for some integer k_0 . The bounds (3.18) follow from (3.16), (3.17), and the following result.

PROPOSITION 3.5. *For any $\delta > 0$ there exists a $K \in \mathbb{N}$ such that*

$$\sum_{k=K}^{\infty} \int_0^1 d\mu_\varepsilon^k \leq \delta \quad \forall \varepsilon.$$

Proof. Notice that

$$(3.19) \quad \int_{\tilde{\Omega}_\varepsilon^0} p U_\varepsilon' \overline{U_\varepsilon'} = N_\varepsilon \sum_{k,l=1}^{\infty} \sum_{j=0}^{N_\varepsilon-1} \int_{Q_0} p(x) \hat{U}_\varepsilon^k\left(\frac{j}{N_\varepsilon}\right) (v^k)' \left(\frac{j}{N_\varepsilon}, x\right) \overline{\hat{U}_\varepsilon^l\left(\frac{j}{N_\varepsilon}\right) (v^l)' \left(\frac{j}{N_\varepsilon}, x\right)} dx$$

$$= N_\varepsilon \sum_{k=1}^{\infty} \sum_{j=0}^{N_\varepsilon-1} \lambda_k\left(\frac{j}{N_\varepsilon}\right) \left| \hat{U}_\varepsilon^k\left(\frac{j}{N_\varepsilon}\right) \right|^2 = \sum_{k=1}^{\infty} \int_0^1 \lambda_k d\mu_\varepsilon^k.$$

In the three equalities of (3.19) we use (3.14), (3.16), and the fact that $v^k(\theta)$ are orthogonal eigenfunctions of $A(\theta)$ with eigenvalues $\lambda_k(\theta)$; see section 2. Combining this observation with (3.12), we find that

$$(3.20) \quad \|p\|_{L^\infty(Q)} \widehat{C}^2 \geq \int_{\tilde{\Omega}^\varepsilon} p |U_\varepsilon'|^2 = \sum_{k=1}^{\infty} \int_0^1 \lambda_k d\mu_\varepsilon^k,$$

where we use the fact that $U_\varepsilon' = 0$ on $F_1 \cap N_\varepsilon Q$. For any integer $K \geq 2$ the inequality (3.20) immediately implies

$$\sum_{k=K}^{\infty} \int_0^1 d\mu_\varepsilon^k \leq \|p\|_{L^\infty(Q)} \widehat{C}^2 \min_{\theta \in [0,1]} \lambda_K(\theta)^{-1} \xrightarrow{K \rightarrow \infty} 0.$$

This concludes the proof of the proposition. \square

Step 4: We show next that $\lambda = \lambda_{k_0}(\theta)$ for some $k_0 \in \mathbb{N}$, $\theta \in [0, 1)$. To this end, for all ε and continuous ψ such that $|\psi| \leq 1$, we first consider a “modulation” (see [2]) $\mathcal{U}_\varepsilon \in V_{N_\varepsilon Q}$ of the function U_ε , given by

$$\mathcal{U}_\varepsilon(x) = \sum_{j=0}^{N_\varepsilon-1} \psi\left(\frac{j}{N_\varepsilon}\right) \sum_{k=1}^{\infty} \hat{U}_\varepsilon^k\left(\frac{j}{N_\varepsilon}\right) v^k\left(\frac{j}{N_\varepsilon}, x\right).$$

Notice that due to the bound on ψ and the estimates (3.12), (3.13) one has

$$(3.21) \quad \|\mathcal{U}_\varepsilon\|_{L^2(N_\varepsilon Q)} \leq \|U_\varepsilon\|_{L^2(N_\varepsilon Q)} \leq 1 + (\varepsilon + c_0 q_\varepsilon^{-1}) C_\perp C_B,$$

$$(3.22) \quad \|\mathcal{U}'_\varepsilon\|_{L^2(N_\varepsilon Q)} \leq \hat{C} \|p\|_{L^\infty(Q)} \|p^{-1}\|_{L^\infty(Q)},$$

where the second estimate is obtained by repeating the argument of (3.19) with U_ε replaced by \mathcal{U}_ε .

Further, we observe that $\varphi_\varepsilon = \chi_\varepsilon \mathcal{U}_\varepsilon \in \tilde{\mathfrak{H}}$ since $H_0^1(\Omega) \subset \mathfrak{H}$, and we choose φ_ε as a test function in the weak formulation (3.7):

$$(3.23) \quad \varepsilon^{-2} \int_{\tilde{\Omega}_1^\varepsilon} p v'_\varepsilon \overline{(\chi_\varepsilon \mathcal{U}_\varepsilon)'} + \int_{\tilde{\Omega}_0^\varepsilon} p v'_\varepsilon \overline{(\chi_\varepsilon \mathcal{U}_\varepsilon)'} = \lambda_\varepsilon \int_{\tilde{\Omega}_\varepsilon} v_\varepsilon \overline{\chi_\varepsilon \mathcal{U}_\varepsilon}.$$

Notice that by construction $\chi_\varepsilon \in V_{N_\varepsilon}$; hence $\chi_\varepsilon \mathcal{U}_\varepsilon$ is piecewise constant in $\tilde{\Omega}_1^\varepsilon$, and therefore the first term in (3.23) vanishes. We also have

$$\int_{\tilde{\Omega}_0^\varepsilon} p v'_\varepsilon \overline{(\chi_\varepsilon \mathcal{U}_\varepsilon)'} = \int_{\tilde{\Omega}_0^\varepsilon} p (\chi_\varepsilon v_\varepsilon)' \overline{\mathcal{U}'_\varepsilon} + o(1) \quad \text{as } \varepsilon \rightarrow 0,$$

since

$$v'_\varepsilon \overline{(\chi_\varepsilon \mathcal{U}_\varepsilon)'} = (\chi_\varepsilon v_\varepsilon)' \overline{\mathcal{U}'_\varepsilon} + v'_\varepsilon \chi'_\varepsilon \overline{\mathcal{U}'_\varepsilon} - v_\varepsilon \chi'_\varepsilon \overline{\mathcal{U}'_\varepsilon},$$

and by virtue of (3.21)–(3.22), the fact that $\|\chi'_\varepsilon\|_{L^\infty(N_\varepsilon Q)} \rightarrow 0$ as $\varepsilon \rightarrow 0$, and the a priori bounds (3.8), we infer that

$$\begin{aligned} & \int_{\tilde{\Omega}_0^\varepsilon} p (v'_\varepsilon \chi'_\varepsilon \overline{\mathcal{U}'_\varepsilon} - v_\varepsilon \chi'_\varepsilon \overline{\mathcal{U}'_\varepsilon}) \\ & \leq \|p\|_{L^\infty(Q)} \|\chi'_\varepsilon\|_{L^\infty(N_\varepsilon Q)} \left(\|v'_\varepsilon\|_{L^2(\tilde{\Omega}_0^\varepsilon)} \|\mathcal{U}_\varepsilon\|_{L^2(\tilde{\Omega}_0^\varepsilon)} + \|\mathcal{U}'_\varepsilon\|_{L^2(\tilde{\Omega}_0^\varepsilon)} \|v_\varepsilon\|_{L^2(\tilde{\Omega}_0^\varepsilon)} \right) \\ & \leq C \|\chi'_\varepsilon\|_{L^\infty(N_\varepsilon Q)} \rightarrow 0, \quad \varepsilon \rightarrow 0. \end{aligned}$$

Next, since $\chi_\varepsilon v_\varepsilon = U_\varepsilon + V_\varepsilon$ and in view of (3.11), (3.22), we get

$$\int_{\tilde{\Omega}_0^\varepsilon} p (\chi_\varepsilon v_\varepsilon)' \overline{\mathcal{U}'_\varepsilon} = \int_{\tilde{\Omega}_0^\varepsilon} p U'_\varepsilon \overline{\mathcal{U}'_\varepsilon} + o(1), \quad \varepsilon \rightarrow 0.$$

Furthermore, recalling the argument of (3.19), we have

$$(3.24) \quad \int_{\tilde{\Omega}_0^\varepsilon} p U_\varepsilon' \overline{\mathcal{U}_\varepsilon} = \sum_{k=1}^{\infty} \int_0^1 \lambda_k \psi d\mu_\varepsilon^k.$$

Similarly we show that

$$(3.25) \quad \int_{\tilde{\Omega}^\varepsilon} v_\varepsilon \overline{\chi_\varepsilon \mathcal{U}_\varepsilon} = \sum_{k=1}^{\infty} \int_0^1 \psi d\mu_\varepsilon^k + o(1), \quad \varepsilon \rightarrow 0.$$

Combining (3.23), (3.24), (3.25) yields

$$(3.26) \quad \sum_{k=1}^{\infty} \int_0^1 \lambda_k \psi d\mu_\varepsilon^k = \lambda_\varepsilon \sum_{k=1}^{\infty} \int_0^1 \psi d\mu_\varepsilon^k + o(1), \quad \varepsilon \rightarrow 0.$$

Finally, passing to the limit $\varepsilon \rightarrow 0$ in (3.26) and using the fact that ψ and λ_k , $k = 1, 2, \dots$, are all continuous functions of θ (see Appendix B) yields

$$\sum_{k=1}^{\infty} \int_0^1 \lambda_k \psi d\mu^k = \lambda \sum_{k=1}^{\infty} \int_0^1 \psi d\mu^k.$$

Using again the fact that λ_k , $k = 1, 2, \dots$, are continuous and recalling that μ^{k_0} is a positive measure for some k_0 , we argue that $\lambda = \lambda_{k_0}(\theta)$ for some $\theta \in [0, 1]$, k_0 , due to the arbitrary choice of the modulating function ψ . \square

4. An example of a family with $\lim_{\varepsilon \rightarrow 0} \mathcal{S}^\varepsilon = \sigma_{\text{Bloch}}$. Here we consider the case when the endpoints of the material domain belong to the “stiff” component. We also assume for simplicity that $(a, b) = (0, 1)$ and restrict our analysis to values $\varepsilon = N^{-1}$, where $N \in \mathbb{N}$; hence $(\varepsilon^{-1}a, \varepsilon^{-1}b) = (0, N)$. Recall that F_0 denotes the 1-periodic extension of $Q_0 = (\alpha, \beta)$ to the whole of \mathbb{R} and $F_1 = \mathbb{R} \setminus F_0$. For all $N \in \mathbb{N}$ we denote by W_N , W_N^\perp the subspace

$$W_N := \{u \in H^1(0, N) \mid u' = 0 \text{ on } (0, N) \cap F_1\}$$

and its orthogonal complement in $H^1(0, N)$, respectively. We also denote by $P_{W_N^\perp}$ the orthogonal projection of $H^1(0, N)$ onto W_N^\perp .

LEMMA 4.1. *There exists a constant $C > 0$ independent of N such that*

$$(4.1) \quad \|P_{W_N^\perp} u\|_{H^1(0, N)} \leq C \|u'\|_{L^2((0, N) \cap F_1)} \quad \forall u \in H^1(0, N).$$

Proof. Recall (see Proposition 3.4) that there exists $C_\perp > 0$ independent of N such that

$$\|P_{V_N^\perp} u\|_{H^1(0, N)} \leq C_\perp \|u'\|_{L^2((0, N) \cap F_1)} \quad \forall u \in H_\#^1(0, N),$$

where P_{V_N} is the orthogonal projection of $H_\#^1(0, N)$ onto

$$V_N := \{u \in H_\#^1(0, N) \mid u' = 0 \text{ on } F_1\}.$$

In what follows we use this fact to prove the desired estimate (4.1). Take χ to be a smooth cut-off function such that $\chi = 1$ on $(0, \alpha/2) \cup (N - (1 - \beta)/2, N)$, $\chi = 0$ on

$[\alpha, N - 1 + \beta]$, and the inequality $|\chi'| \leq \max\{4/\alpha, 4/(1 - \beta)\}$ holds. For fixed u , the function $w := u - \chi u$ is seen to belong to $H^1_{\#}(0, N)$ and satisfy $u - P_{V_N} w = P_{V_N^\perp} w + \chi u$. Therefore, since $V_N \subset W_N$, one has

$$\begin{aligned} \|P_{W_N^\perp} u\|_{H^1(0, N)} &= \inf_{v \in W_N} \|u - v\|_{H^1(0, N)} \leq \|u - P_{V_N} w\|_{H^1(0, N)} \\ &= \|P_{V_N^\perp} w + \chi u\|_{H^1(0, N)} \leq C_\perp \|w'\|_{L^2((0, N) \cap F_1)} + \|\chi u\|_{H^1(0, N)} \\ &\leq (C_\perp + 1)(\|u'\|_{L^2((0, N) \cap F_1)} + \|\chi u\|_{H^1(0, N)}). \end{aligned}$$

It remains to prove the required bound for $\|\chi u\|_{H^1(0, N)}$. By noting the formulae

$$(\chi u)(x) = - \int_x^\alpha (\chi u)', \quad x \in (0, \alpha), \quad \text{and} \quad (\chi u)(x) = \int_{N-1+\beta}^x (\chi u)', \quad x \in (N-1+\beta, N),$$

we find that

$$\int_0^N |\chi u|^2 \leq (1 - \beta + \alpha) \int_0^N |(\chi u)'|^2$$

and

$$\int_0^N |(\chi u)'|^2 \leq \max\{4/\alpha, 4/(1 - \beta)\} \left(\int_0^\alpha + \int_{N-1+\beta}^N \right) |u|^2 + \left(\int_0^\alpha + \int_{N-1+\beta}^N \right) |u'|^2.$$

The result now follows from the formulae

$$u(x) = \int_0^x u' + u(0), \quad x \in (0, \alpha), \quad \text{and} \quad u(x) = - \int_x^N u' + u(N), \quad x \in (N-1+\beta, N),$$

and the fact that the space $H^1(0, N)$ is continuously embedded in $C[0, N]$, which provides suitable bounds on $u(0)$, $u(N)$. \square

We now consider a sequence of λ_ε such that $\lambda_\varepsilon \rightarrow \lambda$ as $\varepsilon \rightarrow 0$, $\varepsilon^{-1} \in \mathbb{N}$, for some $\lambda \in \mathbb{R}$, and the corresponding eigenfunction sequence $v_\varepsilon \in \mathfrak{H}$ satisfies the identity

$$(4.2) \quad \int_0^N (\varepsilon^{-2} \chi_1 + \chi_0) p v'_\varepsilon \varphi' = \lambda_\varepsilon \int_0^N v_\varepsilon \varphi \quad \forall \varphi \in \mathfrak{H},$$

for all values of ε from the indicated set. Here, as before, \mathfrak{H} is a subspace of $H^1(0, N)$ and χ_0 , χ_1 are the characteristic functions of the sets F_0 , F_1 .

Notice first that (4.2) in combination with Lemma 4.1 implies the existence of $C > 0$ such that

$$\|P_{W_N^\perp} v_\varepsilon\|_{L^2(0, N)} \leq C\varepsilon \|v_\varepsilon\|_{L^2(0, N)}.$$

Further, the function $u_\varepsilon = P_{W_N} v_\varepsilon$ is not periodic in general; however, it is piecewise constant on $(0, N) \cap F_1$. We introduce an extension \tilde{u}_ε by the formula

$$(4.3) \quad \tilde{u}_\varepsilon(x) = \begin{cases} u_\varepsilon(x), & x \in (0, N), \\ u_\varepsilon(2N - x), & x \in (N, 2N). \end{cases}$$

Clearly, the function \tilde{u}_ε is $2N$ -periodic, but it is not necessarily an element of the space V_{2N} . However, in cases when \tilde{u}_ε does belong to V_{2N} , we can proceed with the Bloch measure argument in the proof of Theorem 3.1 to show that $\lambda \in \sigma_{\text{Bloch}}$.

As the mapping $y \mapsto 2N - y$ takes intervals of the form $(n, n + a)$ to intervals $(2N - n - \alpha, 2N - n)$ and $(n + \beta, n + 1)$ to $(2N - n - 1, 2N - n - \beta)$, we find that \tilde{u}_ε belongs to the space V_{2N} if

$$2N - n - (2N - n - \alpha) \geq 1 - \beta \quad \text{and} \quad 2N - n - \beta - (2N - n - 1) \geq \alpha,$$

e.g., $\alpha \geq 1 - \beta$ and $1 - \beta \geq \alpha$, which implies $1 - \beta = \alpha$. Hence, the following theorem holds.

THEOREM 4.2. *Assume the conditions described at the beginning of this section hold, and suppose in addition that $\alpha = 1 - \beta$. Then the limit spectrum for sequences with $\varepsilon^{-1} \in \mathbb{N}$ coincides with the Bloch spectrum σ_{Bloch} .*

5. A modified problem with a compact perturbation, and the associated defect modes.

5.1. Analytical set-up. In this section we discuss a modified version of the set-up of section 1.2, as follows. Consider the operator \tilde{A}^ε defined by the sesquilinear form

$$(5.1) \quad (\tilde{A}^\varepsilon u, v) = \int_a^b \left(p_d \chi_d(x) + p(x/\varepsilon)(\varepsilon^2 \chi_0(x/\varepsilon) + \chi_1(x/\varepsilon))(1 - \chi_d(x)) \right) u'(x) \overline{v'(x)} dx,$$

$u, v \in \mathfrak{H}$. Here the space \mathfrak{H} is as before (see section 1.2), and χ_d is the indicator function of a “defect” interval I_d whose closure is assumed to be contained in (a, b) , and p_d is the corresponding “defect” coefficient (or “defect strength”). We denote by \tilde{S}^ε the spectrum of the operator \tilde{A}^ε .

A formal two-scale asymptotic procedure carried out on the equation (cf. (1.1)) $\tilde{A}^\varepsilon u = \lambda u$ suggests the following.

(1) The set $\lim_{\varepsilon \rightarrow 0} \tilde{S}^\varepsilon$ is given by the union of the “Bloch spectrum” given by $\sigma_{\text{Bloch}} = \bigcup_{\theta \in [0, 1)} \sigma(A(\theta))$, the “boundary spectrum” σ_{boundary} (see (3.1)), and a sequence of “defect eigenvalues” $\{p_d \pi^2 j^2 / |I_d|^2\}_{j=1}^\infty$.

(2) Those defect eigenfunctions u_j , $j \in \mathbb{N}$, that correspond to the defect eigenvalues situated in the gaps of σ_{Bloch} decay exponentially away from the boundary of the defect:

$$(5.2) \quad |u_j(x)| \leq c_1 \exp(-c_2 \varepsilon^{-1} \text{dist}\{x, I_d\}), \quad x \in (a, b) \setminus I_d, \quad c_1, c_2 > 0.$$

Here c_2 depends on the distance of the defect eigenvalue to σ_{Bloch} .

In section 5.2 we present numerical evidence that supports these claims. Note that the rate of decay in the estimate (5.2) increases as $\varepsilon \rightarrow 0$. This complements the recent results of [5], where the decay of eigenfunctions in multidimensional high-contrast problems is analyzed. It was shown in [5] that under the assumption that the matrix of the composite forms a connected set in the whole space the defect eigenfunctions corresponding to eigenvalues in the gaps of the limit spectrum satisfy the estimate $|u_j(x)| \leq c_1 \exp(-c_2 \text{dist}\{x, \partial D\})$, $x \in (a, b) \setminus D$, with $c_1, c_2 > 0$, where D is the defect set, and c_2 may depend on j . The higher rate of decay in (5.2) can be interpreted as a stronger localization effect in the case when the matrix of the composite is disconnected.

5.2. Numerical results for the modified problem. We consider a defect of length $|I_d| = 1/2$ and strength $p_d = 2$ in the middle of the interval $(a, b) = (0, 1)$, e.g., $I_d = (1/4, 3/4)$. For each value of ε such that $N := (4\varepsilon)^{-1}$ is a positive integer, we describe the intervals $(0, 1/4)$ and $(3/4, 1)$ on either side of the defect according to (5.1): each of them consists of N cells of the same length $\varepsilon = 1/(4N)$, and in one half of each cell the coefficient in the form (the “modulus”) takes the value $1/(4N)^2$, while in the other half it is equal to unity. Note that the described set-up satisfies the conditions of section 4, so in this case we expect $\lim_{\varepsilon \rightarrow 0} S^\varepsilon$ to coincide with the union of σ_{Bloch} and the sequence of defect eigenvalues.

The results of solving the above problems with finite elements are given in Tables 1 and 2. The values for the trapped mode are in good agreement with the values obtained by the asymptotic method: $\lambda_\star^{(2)} = 78.9568$, $\lambda_\star^{(3)} = 315.8273$, $\lambda_\star^{(5)} = 710.6115$, $\lambda_\star^{(6)} = 1263.3094$, where the superscript is the number of the stop band containing the related eigenvalue.

In addition, the profiles obtained for such trapped modes (see Figure 2 for the case of periodic boundary conditions) suggest that the number of half-oscillations in a trapped mode is equal to the number of the mode in the sequence, which resembles the behavior of the usual Neumann eigenfunctions on the defect. We also note that the decay of the trapped modes appears to be exponential, as can be seen in Figure 3: the larger the contrast (and hence the number of subdivisions of the string), the more localized the mode, irrespective of the boundary conditions at the endpoints of the string.

TABLE 1

Stop bands and trapped modes for the modified problem with a defect, subject to the Dirichlet boundary conditions: $\lambda_{\min}^{(k,128)}$ and $\lambda_{\max}^{(k,128)}$ are the lower and upper bounds of the k th stop band for $N = 128$, and $\lambda_\star^{(k,128)}$, $\lambda_\star^{(k,256)}$, $\lambda_\star^{(k,512)}$, $\lambda_\star^{(k,1024)}$ are the trapped-mode eigenvalues in the k th stop band, evaluated for $N = 128$, $N = 256$, $N = 512$, and $N = 1024$, respectively.

Dirichlet boundary conditions					
$\lambda_{\min}^{(k,128)}$	$\lambda_{\max}^{(k,128)}$	$\lambda_\star^{(k,128)}$	$\lambda_\star^{(k,256)}$	$\lambda_\star^{(k,512)}$	$\lambda_\star^{(k,1024)}$
11.7939	39.4603	-	-	-	-
65.7875	157.8859	75.7674	77.2502	78.0741	78.7304
187.6799	355.2599	293.9534	304.1141	309.7163	314.2461
386.1413	622.2747	-	-	-	-
662.9213	986.7685	682.6577	694.4984	702.0486	708.4576
1018.4394	1421.0468	1225.1298	1232.2190	1243.1182	1258.2799

TABLE 2

Stop bands and trapped modes for the modified problem with a defect, subject to the Neumann boundary conditions: $\lambda_{\min}^{(k,128)}$ and $\lambda_{\max}^{(k,128)}$ are the lower and upper bounds of the k th stop band for $N = 128$, and $\lambda_\star^{(k,128)}$, $\lambda_\star^{(k,256)}$, $\lambda_\star^{(k,512)}$, $\lambda_\star^{(k,1024)}$ are the trapped-mode eigenvalues in the k th stop band, evaluated for $N = 128$, $N = 256$, $N = 512$, and $N = 1024$, respectively.

Neumann boundary conditions					
$\lambda_{\min}^{(k,128)}$	$\lambda_{\max}^{(k,128)}$	$\lambda_\star^{(k,128)}$	$\lambda_\star^{(k,256)}$	$\lambda_\star^{(k,512)}$	$\lambda_\star^{(k,1024)}$
11.7515	39.4980	-	-	-	-
65.8359	157.8901	75.7676	77.2509	78.0741	78.7314
187.7334	355.2765	293.9539	304.1145	309.7164	314.3057
386.2091	622.2747	-	-	-	-
662.9779	986.7698	682.6578	694.4985	702.0486	708.6496
1018.5163	1421.0556	1225.1298	1232.2190	1243.1193	1258.2626

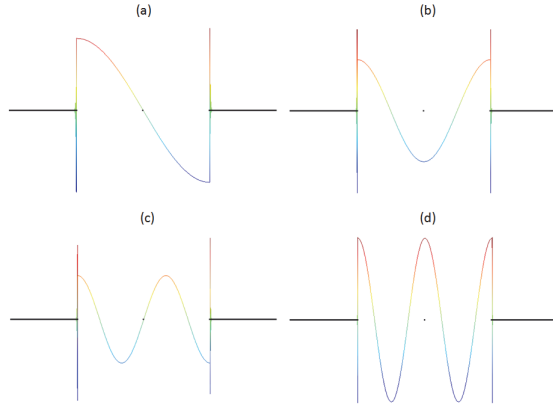


FIG. 2. Trapped eigenmodes for the modified problem with a defect and periodic boundary conditions ($N = 1024$), corresponding to eigenfrequencies: (a) $\lambda_{\star}^{(2,1024)} = 78.07$, (b) $\lambda_{\star}^{(3,1024)} = 314.24$, (c) $\lambda_{\star}^{(5,1024)} = 708.46$, (d) $\lambda_{\star}^{(6,1024)} = 1258.28$.

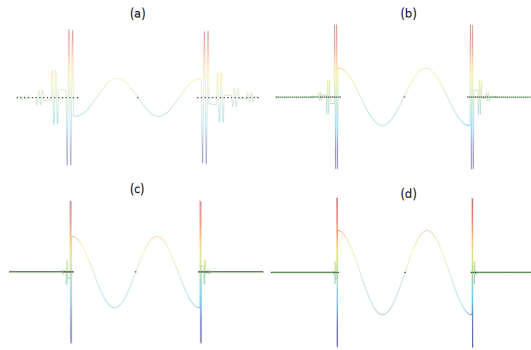


FIG. 3. Decay of the third trapped eigenmode (located in the fifth stop band) for the modified problem with a defect and periodic boundary conditions, as a function of contrast: (a) $\lambda_{\star}^{(5,32)} = 668.86$, (b) $\lambda_{\star}^{(5,64)} = 682.89$, (c) $\lambda_{\star}^{(5,128)} = 694.50$, (d) $\lambda_{\star}^{(5,256)} = 702.68$.

5.3. Photonic band gaps and trapped modes in high-contrast multilayered dielectric structures. The string problem emerges, among other contexts, in the study of wave propagation in one-dimensional photonic crystals, e.g., multilayered dielectric structures invariant along two directions. In what follows, we set these directions to be x_1 and x_3 in the usual Euclidean representation $x = (x_1, x_2, x_3)$.

We consider solutions $(\mathcal{E}, \mathcal{H})$ to the classical system of Maxwell's equations [10] that have the form

$$\mathcal{E}(x_1, x_2, x_3, t) = E(x_2) \exp(i(\kappa x_3 - \omega t)), \quad \mathcal{H}(x_1, x_2, x_3, t) = H(x_2) \exp(i(\kappa x_3 - \omega t)),$$

where t is time, ω is the angular frequency, and $\kappa \geq 0$ is a "propagation constant." We write the Maxwell's equations for the field variable (E, H) :

$$\begin{cases} E'_3 - i\kappa E_2 & = i\omega\mu H_1, \\ i\kappa E_1 & = i\omega\mu H_2, \\ -E'_1 & = i\omega\mu H_3, \end{cases} \quad \begin{cases} -H'_3 + i\kappa H_2 & = i\omega\varepsilon E_1, \\ -i\kappa H_1 & = i\omega\varepsilon E_2, \\ H'_1 & = i\omega\varepsilon E_3. \end{cases}$$

Here μ is the magnetic permeability, and ε is the electric permittivity at each point of the dielectric.

We rearrange the above six equations into two groups of equations for (E_1, H_2, H_3) (transverse magnetic polarization), and (H_1, E_2, E_3) (transverse electric polarization). We choose E_1 and H_1 as the unknown functions within the respective groups and notice that the remaining unknowns are expressed in terms of these two scalar functions only. The equations satisfied by E_1, H_1 are

$$(5.3) \quad (E_1')' + (\omega^2 \mu \varepsilon - \kappa^2) E_1 = 0,$$

$$(5.4) \quad (\varepsilon^{-1} H_1')' + (\omega^2 \mu - \varepsilon^{-1} \kappa^2) H_1 = 0.$$

Note that (5.4) coincides with (1.1) when $\kappa = 0$ by setting

$$\omega^2 = \lambda, \quad \mu = 1, \quad \varepsilon^{-1}(x_2) = p(x_2/\eta)(\eta^2 \chi_0(x_2/\eta) + \chi_1(x_2/\eta)), \quad x_2 \in (a, b), \quad \eta > 0,$$

where we use η rather than ε to denote the structure period in order to avoid confusion with the standard notation for electric permittivity. Our analysis in sections 2 and 3 carries over to the case $\kappa > 0$. In particular, for $p \equiv 1$ we get a κ -dependent version of the dispersion relation (2.1), as follows:

$$\frac{1}{2}(\alpha - \beta + 1) \left(\sqrt{\lambda} - \frac{\kappa^2}{\sqrt{\lambda}} \right) \sin(\sqrt{\lambda}(\alpha - \beta)) + \cos(\sqrt{\lambda}(\alpha - \beta)) = \cos(2\pi\theta).$$

Assuming infinitely conducting walls on either side of the dielectric (see, e.g., [21] for further details), we supply (5.3) and (5.4) with homogeneous Dirichlet and Neumann boundary conditions, respectively.

In the numerical solution of the above problem we employ finite elements with perfectly matched layers, e.g., anisotropic absorptive reflectionless layers (see, e.g., [21]), on the top and bottom of the computational domain. Our results are shown in Figure 4 for $\kappa = 0.1$, $N = 16$, and for the transverse electric mode with frequency $\lambda_* = 78.34$ inside the third stop band. The latter corresponds to the first trapped mode shown in Figure 2(a), in view of the fact that for $\kappa = 0.1$ there is an additional zero-frequency stop band. The magnetic component of this mode (Figures 4(a) and (d)) clearly shares the same features as the string mode in Figure 2(a).

Appendix A. In this appendix we argue that for any fixed $N \in \mathbb{N}$ and $\lambda < 0$, say, $\lambda = -1$, the solutions to the problems (1.1) converge in an appropriate two-scale sense (see, e.g., [1]) to the solution of some limit problem parametrized by N . We first formulate the related statement for $N = 1$.

A.1. Periodic homogenization.

LEMMA A.1. *Set $\lambda = -1$, and let $u_\varepsilon \in \mathfrak{X}$ be the solution to the problem (1.1). Define a space V_1 by*

$$V_1 := \{v \in H_{\#}^1(Q) : v'(y) = 0 \text{ for } y \in F_1\}.$$

There exists $u(x, y) \in L^2((a, b); V_1)$ such that up to a subsequence, which we do not relabel,

$$u_\varepsilon \xrightarrow{2} u(x, y), \quad \varepsilon u_\varepsilon' \xrightarrow{2} \frac{\partial u}{\partial y}(x, y), \quad \chi_1\left(\frac{x}{\varepsilon}\right) p\left(\frac{x}{\varepsilon}\right) u_\varepsilon'(x) \xrightarrow{2} 0,$$

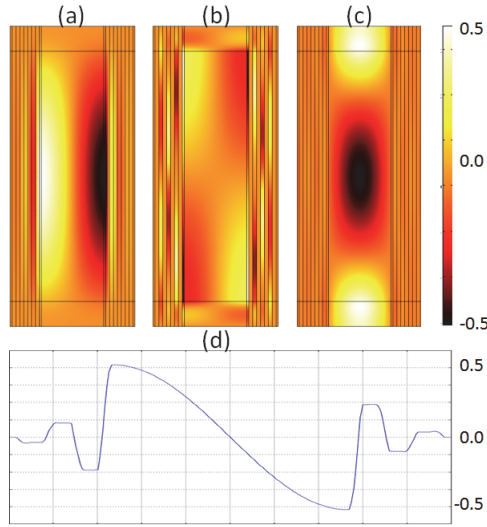


FIG. 4. Obliquely propagating transverse electric wave in a high-contrast dielectric multilayered planar waveguide with infinitely conducting walls: (a) two-dimensional plot of H_1 ; (b) two-dimensional plot of E_2 ; (c) two-dimensional plot of E_3 ; (d) profile of H_1 along the horizontal centerline. Here $N = 16$, $\kappa = 0.1$, and the normalized frequency $\lambda_* = 78.34$.

where “ $\overset{2}{\rightharpoonup}$ ” is weak two-scale convergence (see [1]). The function $u \in L^2((a, b); V_1)$ is the unique solution to the problem

$$(A.1) \quad \int_a^b \int_Q p(y) \chi_0(y) \frac{\partial u}{\partial y}(x, y) \overline{\frac{\partial \varphi}{\partial y}(x, y)} \, dy dx + \int_a^b \int_Q u(x, y) \overline{\varphi(x, y)} \, dy dx = \int_a^b \int_Q f(x) \overline{\varphi(x, y)} \, dy dx \quad \forall \varphi \in C_0^\infty((a, b); V_1).$$

Proof. Step 1. Let u_ε be a solution to problem (1.1), e.g.,

$$(A.2) \quad \int_a^b p\left(\frac{x}{\varepsilon}\right) \left(\varepsilon^2 \chi_0\left(\frac{x}{\varepsilon}\right) + \chi_1\left(\frac{x}{\varepsilon}\right)\right) u'_\varepsilon(x) \overline{\varphi'(x)} \, dx + \int_a^b u_\varepsilon(x) \overline{\varphi(x)} \, dx = \int_a^b f(x) \overline{\varphi(x)} \, dx$$

for all $\varphi \in \mathfrak{H}$. It is clear, by choosing $\varphi = u_\varepsilon$, that the sequences u_ε and $\varepsilon u'_\varepsilon$ are bounded in $L^2(a, b)$. Therefore, by [1, Proposition 1.14] there exists $u(x, y) \in L^2((a, b); H_{\#}^1(Q))$ such that up to a subsequence, $u_\varepsilon \overset{2}{\rightharpoonup} u$ and $\varepsilon u'_\varepsilon \overset{2}{\rightharpoonup} \frac{\partial u}{\partial y}$. We now show that, in fact, $u(x, y) \in L^2((a, b); V_1)$. In view of the convergence $\varepsilon u'_\varepsilon \overset{2}{\rightharpoonup} \frac{\partial u}{\partial y}$, we have

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \int_a^b p\left(\frac{x}{\varepsilon}\right) \chi_1\left(\frac{x}{\varepsilon}\right) u'_\varepsilon(x) \overline{\Psi\left(x, \frac{x}{\varepsilon}\right)} \, dx = \int_a^b \int_Q p(y) \chi_1(y) \frac{\partial u}{\partial y}(x, y) \overline{\Psi(x, y)} \, dy dx \quad \forall \Psi \in L^2((a, b); C_{\#}^\infty(Q)).$$

Furthermore, since the sequence $\chi_1 u'_\varepsilon$ is bounded in $L^2(a, b)$, we have

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \int_a^b p\left(\frac{x}{\varepsilon}\right) \chi_1\left(\frac{x}{\varepsilon}\right) u'_\varepsilon \overline{\Psi\left(x, \frac{x}{\varepsilon}\right)} \, dx = 0.$$

Step 2. Here we show that $\chi_1\left(\frac{x}{\varepsilon}\right) p\left(\frac{x}{\varepsilon}\right) u'_\varepsilon(x) \xrightarrow{2} 0$. Since $\chi_1\left(\frac{x}{\varepsilon}\right) u'_\varepsilon(x)$ is bounded in $L^2(a, b)$, there exists $\xi(x, y) \in L^2((a, b) \times Q)$ such that up to a subsequence, $\chi_1\left(\frac{x}{\varepsilon}\right) u'_\varepsilon(x) \xrightarrow{2} \xi(x, y)$. Therefore, it suffices to show that $p(y)\chi_1(y)\xi(x, y) \equiv 0$. To this end, notice that, taking $\varphi_\varepsilon(x) = \varepsilon\varphi(x)\psi\left(\frac{x}{\varepsilon}\right)$, $\varphi \in C_0^\infty(a, b)$, $\psi(y) \in C_\#^\infty(Q)$ as test functions in (A.2), we obtain

$$\lim_{\varepsilon \rightarrow 0} \int_a^b p\left(\frac{x}{\varepsilon}\right) \chi_1\left(\frac{x}{\varepsilon}\right) u'_\varepsilon(x) \overline{\varphi'_\varepsilon(x)} dx = 0.$$

Notice also that since $\chi_1\left(\frac{x}{\varepsilon}\right) u'_\varepsilon(x) \xrightarrow{2} \xi(x, y)$, one has

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_a^b p\left(\frac{x}{\varepsilon}\right) \chi_1\left(\frac{x}{\varepsilon}\right) u'_\varepsilon(x) \overline{\varphi'_\varepsilon(x)} dx &= \lim_{\varepsilon \rightarrow 0} \int_a^b p\left(\frac{x}{\varepsilon}\right) \chi_1^2\left(\frac{x}{\varepsilon}\right) u'_\varepsilon(x) \overline{(\varepsilon\varphi'(x)\psi\left(\frac{x}{\varepsilon}\right) + \varphi(x)\psi'\left(\frac{x}{\varepsilon}\right))} dx \\ &= \int_a^b \int_Q p(y)\chi_1(y)\xi(x, y) \overline{\varphi(x)\psi'(y)} dy dx. \end{aligned}$$

It follows that

$$(A.3) \quad \int_Q p(y)\chi_1(y)\xi(x, y) \overline{\psi'(y)} dy = 0 \quad \text{for a.e. } x \in (a, b), \forall \psi \in C_\#^\infty(Q).$$

It remains to recall that for any $v \in C_0^\infty(Q_1)$ there exists $\psi \in C_\#^\infty(Q)$ such that $\psi' = v$ in Q_1 , e.g., $\psi(y) := \int_0^y (v + \tilde{\chi}_0)$, where $\tilde{\chi}_0 \in C_0^\infty(Q_0)$ is such that $\int_0^1 (v + \tilde{\chi}_0) = 0$.

Step 3. In order to show that u solves (A.1), we choose $\varphi_\varepsilon(x) = \varphi\left(x, \frac{x}{\varepsilon}\right)$, where $\varphi(x, y) \in C_0^\infty((a, b); V_1)$, as a test function in (A.2) and pass to the limit $\varepsilon \rightarrow 0$, recalling that u_ε , $\varepsilon u'_\varepsilon$ and $\chi_1\left(\frac{x}{\varepsilon}\right) u'_\varepsilon$ two-scale converge to u , $\frac{\partial u}{\partial y}$, and 0, respectively. Uniqueness is shown in the standard way by observing that the problem (A.1) for $f = 0$ has only the trivial solution. \square

Clearly $C_0^\infty((a, b); V_1)$ is dense in $H := L^2((a, b); V_1)$ with respect to the norm induced by the inner product

$$(u, v)_H := \int_a^b \int_Q p(y)\chi_0(y) \frac{\partial u}{\partial y}(x, y) \overline{\frac{\partial v}{\partial y}(x, y)} dy dx + \int_a^b \int_Q u(x, y) \overline{\varphi(x, y)} dy dx.$$

Therefore, the sesquilinear form

$$\mathcal{A}(u, \varphi) := \int_a^b \int_Q p(y)\chi_0(y) \frac{\partial u}{\partial y}(x, y) \overline{\frac{\partial \varphi}{\partial y}(x, y)} dy dx$$

defines a self-adjoint operator A on H , and Lemma A.1 implies that the operator sequence A^ε converges to A in the strong two-scale resolvent sense; see [18]. This implies, in particular, that $\lim_{\varepsilon \rightarrow 0} S^\varepsilon \supset \sigma(A)$. We are now concerned with getting further information about $\sigma(A)$. To this end, notice that the x -dependence in the formulation (A.1) is trivial; indeed, $u(x, y) = f(x)v(y)$, where $v(y) \in V_1$ is the unique solution to the formulation

$$(A.4) \quad \int_{Q_0} p v' \overline{\varphi'} + \int_Q v \overline{\varphi} = \int_Q \overline{\varphi} \quad \forall \varphi \in V_1.$$

This observation implies, in particular, that the spectra corresponding to (A.1) and (A.4) coincide. Therefore, denoting by A_1 the operator defined by the sesquilinear form

$$\mathfrak{b}_1(u, v) := \int_{Q_0} pu' \overline{v'}, \quad u, v \in V_1,$$

we obtain $\lim_{\varepsilon \rightarrow 0} S^\varepsilon \supset \sigma(A) = \sigma(A_1)$.

A.2. “ NQ -periodic” homogenization. In the argument above we found the two-scale limit operator A_1 by choosing the unit cell Q to be the periodic reference cell and by passing to the two-scale limit in (1.1) as $\varepsilon \rightarrow 0$. Replacing now Q with NQ , $N \in \mathbb{N}$, we obtain an analogue of Lemma A.1, as follows.

LEMMA A.2. *Set $\lambda = -1$, and let u_ε be the solution to (1.1). Then, up to a subsequence, $u_\varepsilon \xrightarrow{2} u_N$, where $u_N(x, y) = f(x)v_N(y)$ and v_N is the unique solution to the problem*

$$\int_{NQ} \chi_0 p v'_N \overline{\varphi'} + \int_{NQ} v_N \overline{\varphi} = \int_{NQ} \overline{\varphi} \quad \forall \varphi \in V_N,$$

where $V_N := \{v \in H^1_{\#}(NQ) : v'(y) = 0 \text{ for } y \in Q_1\}$. Furthermore, the inclusion

$$\lim_{\varepsilon \rightarrow 0} S^\varepsilon \supset \sigma(A_N)$$

holds, where A_N is the operator defined using the sesquilinear form

$$\mathfrak{b}_N(u, v) := \int_{NQ} \chi_0 pu' \overline{v'}, \quad u, v \in V_N.$$

Applying Lemma A.2 for all $N \in \mathbb{N}$ yields

$$\lim_{\varepsilon \rightarrow 0} S^\varepsilon \supset \bigcup_{N \in \mathbb{N}} \sigma(A_N).$$

A.3. Relation to the Bloch spectrum. For each $\theta \in [0, 1)$, $k = 1, 2, \dots$, we denote by $\tilde{v}^k(\theta) = \tilde{v}^k(\theta, y)$ the extension of the eigenfunction $v^k(\theta)$ (see section 2) to the whole space \mathbb{R} such that $\tilde{v}^k(\theta, y) \exp(-2\pi i \theta y)$, $y \in \mathbb{R}$, is Q -periodic. Notice that if $\theta = j/N$ for some integers N, j such that $0 \leq j \leq N-1$, then for all $k = 1, 2, \dots$ the functions $\tilde{v}^k(\theta)$ are NQ -periodic; in particular, $\tilde{v}^k(\theta) \in V_N$. In fact, the functions $\tilde{v}^k(\theta)$, $k = 1, 2, \dots$ are eigenfunctions of the operator A_N . Indeed, for all $\varphi \in V_N$ define the function $\Phi(y) := \sum_{k=0}^{N-1} \varphi(y+k) \exp(-2\pi i \theta k)$, and notice that since $\Phi(y) \in V(\theta)$, one has $(A(\theta)v^k(\theta), \Phi) = \lambda_k(\theta)(v^k(\theta), \Phi)$. Therefore, writing for brevity v in place of both $v^k(\theta)$ and $\tilde{v}^k(\theta)$, we obtain

$$\begin{aligned} \int_{NQ} \chi_0 p v' \overline{\varphi} &= \sum_{l=0}^{N-1} \int_Q \chi_0(y+l) p(y+l) v'(y+l) \overline{\varphi(y+l)} dy = \int_Q \chi_0 p v \overline{\Phi} = \lambda_k(\theta) \int_Q v \overline{\Phi} \\ &= \sum_{l=0}^{N-1} \lambda_k(\theta) \int_Q v(y) \exp(2\pi i \theta l) \overline{\varphi(y+l)} dy = \lambda_k(\theta) \int_{NQ} v \overline{\varphi}. \end{aligned}$$

The above implies that

$$\lim_{\varepsilon \rightarrow 0} S^\varepsilon \supset \bigcup_{N \in \mathbb{N}} \sigma(A_N) \supset \bigcup_{\substack{N \in \mathbb{N} \\ 0 \leq j \leq N-1}} \sigma(A(j/N)).$$

Finally, using the facts that the set of rational numbers j/N is dense in $[0, 1)$ and that the eigenvalues $\lambda = \lambda(\theta)$ are continuous with respect to θ (see Appendix B below) yields

$$\lim_{\varepsilon \rightarrow 0} S^\varepsilon \supset \bigcup_{\theta \in [0, 1)} \sigma(A(\theta)).$$

Appendix B. Here we show that the eigenvalues of the operators $A(\theta)$ defined by (2.3) are continuous in θ . The statement of continuity of $\lambda_k = \lambda_k(\theta)$, $k \in \mathbb{N}$, is not a simple consequence of the continuity of eigenvalues for the usual Floquet–Bloch decomposition. An important distinct feature of our case is the dependence on θ of the operator domain $V(\theta)$. The continuity of $\lambda_k(\theta)$ in θ is therefore shown to be a consequence of the continuity of the spaces $V(\theta)$ with respect to θ ; see Lemma 2.3. In fact, we show that the continuity of $V(\theta)$ leads to the operators $A(\theta)$ being continuous with respect to θ in the norm-resolvent sense, which implies as one particular consequence the continuity of $\lambda_k(\theta)$.

THEOREM B.1. *For any $\theta \in [0, 1)$, let $\theta_n \in [0, 1)$ be such that $\theta_n \rightarrow \theta$ as $n \rightarrow \infty$. Then the sequence $(A(\theta_n) + I)^{-1}$ converges to $(A(\theta) + I)^{-1}$ in the operator norm. In particular, the eigenvalues $\lambda_k(\theta)$, $k \in \mathbb{N}$, of $A(\theta)$ are continuous functions of $\theta \in [0, 1)$, e.g., $\lim_{n \rightarrow \infty} \lambda_k(\theta_n) = \lambda_k(\theta)$.*

An essential ingredient to the proof of Theorem B.1 is the following result.

LEMMA B.2. *For any $\theta \in [0, 1)$, let θ_n be such that $\theta_n \rightarrow \theta$ as $n \rightarrow \infty$ and $f_n, f \in L^2(Q)$ be such that $f_n \rightharpoonup f$ in $L^2(Q)$. Then, the solutions $u_n \in V(\theta_n)$ of the problems $(A(\theta_n) + I)u_n = f_n$ weakly converge in $H^1(Q)$ to the solution $u \in V(\theta)$ of the problem $(A(\theta) + I)u = f$.*

Proof. Let u_n be as in the statement of the lemma; then

$$(B.1) \quad \int_Q pu'_n \bar{\varphi} + \int_Q u_n \bar{\varphi} = \int_Q f_n \bar{\varphi} \quad \forall \varphi \in V(\theta_n).$$

As the sequence f_n is weakly convergent, it is bounded, and we find by choosing $\varphi = u_n$ in (B.1) that $\|u_n\|_{H^1(Q)} \leq \|f_n\|_{L^2(Q)}$; e.g., the sequence u_n is bounded in $H^1(Q)$. In particular, up to a subsequence, u_n converges weakly in $H^1(Q)$ (hence strongly in $L^2(Q)$) to some $u \in H^1(Q)$. It is readily shown that $u \in V(\theta)$.

Furthermore, for a fixed $\varphi \in V(\theta)$ there exist, by Lemma 2.3, $\varphi_n \in V(\theta_n)$ such that $\varphi_n \rightarrow \varphi$ strongly in $H^1(Q)$. Choosing φ_n as the test function in (B.1) and passing to the limit $n \rightarrow \infty$ shows that u is a solution to

$$\int_Q pu' \bar{\varphi} + \int_Q u \bar{\varphi} = \int_Q f \bar{\varphi} \quad \forall \varphi \in V(\theta).$$

By virtue of the fact that the solution u is unique, the above argument applies to any subsequence of u_n . Therefore, the claim holds for the whole sequence u_n . \square

We shall now proceed with the proof of Theorem B.1.

Step 1. Let θ, θ_n satisfy the assumptions of the theorem. First, we show that the operator sequence $R_n := (A(\theta_n) + I)^{-1}$ converges uniformly to $R := (A(\theta) + I)^{-1}$, e.g.,

$$\|R_n - R\| = \sup_{\|f\|_{L^2(Q)}=1} \|R_n f - Rf\|_{L^2(Q)} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

For all n , let f_n be such that

$$\sup_{\|f\|_{L^2(Q)}=1} \|R_n f - Rf\|_{L^2(Q)} \leq \|R_n f_n - Rf_n\|_{L^2(Q)} + \frac{1}{n}.$$

Since $\|f_n\|_{L^2(Q)} = 1$, the sequence f_n has a subsequence that converges weakly to some $f_0 \in L^2(Q)$. Therefore, by Lemma B.2 the sequence $R_n f_n$ converges to Rf_0 strongly in $L^2(Q)$. Furthermore, since R is compact, we infer that Rf_n converges to Rf_0 strongly in $L^2(Q)$, and therefore

$$\sup_{\|f\|_{L^2(Q)}=1} \|R_n f - Rf\|_{L^2(Q)} \leq \|R_n f_n - Rf_0\|_{L^2(Q)} + \|Rf_n - Rf_0\|_{L^2(Q)} + \frac{1}{n}.$$

The right-hand side of the above estimate converges to zero as $n \rightarrow \infty$. The result follows as this argument holds for any subsequence of f_n and therefore for the whole sequence f_n .

Step 2. We shall now show the continuity in θ of the eigenvalues $\lambda_k(\theta)$. To this end we establish that the eigenvalues $\mu_k(\theta) = (\lambda_k(\theta) + 1)^{-1}$ of the operator $(A(\theta) + I)^{-1}$ are continuous in θ . To prove that $\mu_k(\theta)$ are continuous, we note that for any $f \in L^2(Q)$ one has

$$\frac{(R_n f, f)_{L^2(Q)}}{\|f\|_{L^2(Q)}^2} - \|R_n - R\| \leq \frac{(Rf, f)_{L^2(Q)}}{\|f\|_{L^2(Q)}^2} \leq \frac{(R_n f, f)_{L^2(Q)}}{\|f\|_{L^2(Q)}^2} + \|R_n - R\|,$$

and the min-max variational principle (cf. [16])

$$\mu_k(\theta) = \inf_{\substack{F \subset L^2(Q), \\ \dim F = k}} \sup_{f \in F, \|f\|_{L^2(Q)}=1} \frac{(Rf, f)_{L^2(Q)}}{\|f\|_{L^2(Q)}^2}$$

implies that $|\mu_k(\theta_n) - \mu_k(\theta)| \leq \|R_n - R\|$. Invoking the uniform convergence of R_n to R as $n \rightarrow \infty$ proves the result.

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