Travelling Waves in Heterogeneous Media

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Summary

In this thesis we study the existence of travelling wave type solutions for a reaction diffusion equation in $\mathbb{R}^2$ with a nonlinearity which depends periodically on the spatial variable. Specifically we will consider a particular class of nonlinearities where we treat the coefficient of the linear term as a parameter. For this class of nonlinearities we formulate the problem as a spatial dynamical system and use a centre manifold reduction to find conditions on the parameter and nonlinearity for the existence of travelling wave type solutions with particular wave speeds. We then consider what happens if the parameter and the wave speed vary close to zero; by analysing the bifurcations in this case we are able to find travelling wave solutions with periodic and homoclinic structures. Finally we examine what happens to the travelling wave solutions as the period of the periodic dependence in the nonlinearity tends to zero.

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Introduction

1.1 Problem

In this thesis we will investigate the existence of travelling wave type solutions for a reaction diffusion equation in $\mathbb{R}^2$ with a nonlinearity that is periodically dependent on the spatial variable $x \in \mathbb{R}^2$. Specifically, we will consider the equation

$$u_t = \operatorname{div}(A\nabla u) + f\left(\frac{x}{\varepsilon}, u\right), \tag{1.1}$$

where A is a real symmetric positive definite matrix, $\varepsilon > 0$ and the nonlinearity $f = f(\xi, u)$ is periodic in ξ with periodic cell $[0, 2\pi]^2$, i.e.

$$f(\xi_1 + \xi_2, u) = f(\xi_1, u) \text{ for all } \xi_2 \in (2\pi\mathbb{Z})^2 .$$

For this equation we will look for travelling wave type solutions: A solution is a classical travelling wave solution if it has the form

$$u(x, t) = v(x \cdot k - ct), \tag{1.2}$$

where $k \in S^1$, $c \neq 0$ is a constant and $v = v(\tau)$ is a fixed profile. Hence they are solutions which are fixed orthogonal to k and move in the k direction with speed c as time varies. However due to the periodic behaviour in the nonlinearity f such travelling wave solutions will not in general exist for equation (1.1). Therefore we need to modify the type of solution we look for to accommodate this periodic behaviour. Thus we look for generalised travelling wave solutions of the form

$$u(x, t) = v^\varepsilon\left(x \cdot k - ct, \frac{x}{\varepsilon}\right),$$

where the profile function $v^\varepsilon = v^\varepsilon(\tau, \xi)$, with $\tau = x \cdot k - ct$ and $\xi = x/\varepsilon$, is periodic in ξ with periodic cell $[0, 2\pi]^2$. This type of travelling wave solution has a profile which varies as it moves over the periodic cells and therefore is able to incorporate the effect of the periodic dependence of the nonlinearity into the solution.

After we have established the existence of these generalised travelling wave solutions in a variety of different cases, we will then look at a particular case where we have proved existence and examine what happens to these generalised travelling wave solutions as $\varepsilon \to 0$. For this particular case we will show that there exist a limiting profile $v^0(\tau)$ such that

$$v^\varepsilon\left(x \cdot k - ct, \frac{x}{\varepsilon}\right) = v^0\left(x \cdot k - ct\right) + O(\varepsilon) \text{ as } \varepsilon \to 0,$$

which tells us that the generalised travelling wave solution is of order ε close to a fixed profile.

The approach we will use to study the existence of these generalised travelling wave solutions will be to firstly substitute the ansatz (1.2) for these solutions into the reaction diffusion equation (1.1) to get an equation in terms of the profile function v^ε. Once we have this equation in terms of v^ε we will formulate it as a spatial dynamical system and use a local centre manifold reduction to investigate the existence of generalised travelling wave solutions. The limit as $\varepsilon \to 0$ is then studied by rescaling the travelling wave solutions and examining what happens to the local centre manifold as $\varepsilon \to 0$.

1.2 Background

In the previous section we gave an overview of the problems we will deal with in this thesis. Our aim in this section is to put these problems into context by providing some background information on the reaction diffusion equation and outlining the history of travelling wave solutions for the reaction diffusion equation. There exists a vast literature on these two subjects and the references we give in this section only represent a small subset of this literature. For more details see the review article [51] or book [52] by Xin, the paper by Berestycki and Hamel [4] or the book by A. Volpert, V. Volpert and V. Volpert [48], and the references there in.

1.2.1 Reaction Diffusion Equation

We begin by giving some background information on the reaction diffusion equation and how the associated Cauchy problem can be solved.

In this thesis we will investigate a reaction diffusion equation

$$u_t = \operatorname{div}\left(A \nabla u\right) + f\left(\frac{x}{\varepsilon}, u\right).$$

This type of equation models the evolution of a quantity u under the action of diffusion and a local reaction. These two processes appear in a wide range of applications and thus, with appropriate choices of domain, boundary conditions and nonlinearity, reaction diffusion equations appear as models in many different fields. A few examples of fields where reaction diffusion equations or systems are used are biology [2, 11, 19, 20], combustion [37, 49, 55] and nonlinear heat flow [12].

An important question associated with reaction diffusion equations is the existence of solutions for the related Cauchy problem. An example of a Cauchy problem would be to find a function $u(x,t)$ such that

$$\begin{aligned} u_t &= \operatorname{div}\left(A\nabla u\right) + f\left(\frac{x}{\varepsilon}, u\right) \\ u(x,0) &= u_0(x). \end{aligned}$$

This Cauchy problem is well posed on $C_{\text{unif}}\left(\mathbb{R}^2\right)$, the space of uniformly continuous and bounded functions on $\mathbb{R}^2$. This well-posedness can be proved by using semigroup methods. For details of these methods see the books by Henry [28] or Pazy [41].

1.2.2 Travelling Wave Solutions

Now that we have given some background on the reaction diffusion equation and how the associated Cauchy problem can be solved, we will move on to give an overview of the history of travelling solutions for the reaction diffusion equation. The idea of travelling wave solutions where introduced in 1937 in the work of Kolmogorov, Petrovsky and Piskunov [34] and Fisher [20]. In the paper by Kolmogorov, Petrovsky and Piskunov they considered a reaction diffusion equation in $\mathbb{R}$ with a monostable nonlinearity

$$u_t = \frac{1}{2}u_{xx} + f(u)$$

where

$$\begin{cases} f > 0 \text{ on } (0,1) \\ f(0) = f(1) = 0 \end{cases}. \tag{1.3}$$

For this equation they proved the existence of a monotone travelling wave solution

$$u(x,y) = v\left(x \cdot k - ct\right), \tag{1.4}$$

where $v = v(\tau)$ is a fixed profile such that $v(\infty) = 1$ and $v(-\infty) = 0$, for all $c \geqslant c_\star$ the critical wave speed. The existence of these monotone travelling wave solutions was proved by firstly substituting the traveling wave ansatz (1.4) into the reaction diffusion equation to get an ordinary differential equation in terms of the profile function v

$$0 = \frac{1}{2}v_{\tau\tau} + cv_\tau + f(v)$$

and then studying the phase plane associated with this ordinary differential equation.

Subsequently the existence of travelling wave solutions was extended to the cases of combustion type nonlinearities where

$$\begin{cases} \exists\theta \in (0,1) \text{ such that } f(s) = 0 \text{ for all } s \in [0,\theta], \\ f(s) > 0 \text{ for all } s \in (\theta, 1), \\ f(1) = 0 \end{cases} \tag{1.5}$$

and bistable nonlinearities where

$$\begin{cases} \exists\theta \in (0,1) \text{ such that } f(s) < 0 \text{ for all } s \in (0,\theta), \\ f(s) > 0 \text{ for all } s \in (\theta, 1), \\ f(s) = 0 \text{ for all } s \in \{0, \theta, 1\} \end{cases} \tag{1.6}$$

in the work of Kanel′ [31] and [32]. The case of the bistable nonlinearity is slightly different to the monostable and combustion nonlinearities as in addition to travelling wave solutions which connect between 0 and 1 there exist travelling wave profiles which do not connect between 0 and 1. Instead these travelling wave profiles will connect 0 and 1 to the intermediate stationary point θ, detail can be found in Fife [19]. Also the asymptotic behaviour of the critical front, which has speed $c_\star$, was worked out by Aronson and Weinberger [2].

Later the results of Kolmogorov, Petrovsky and Piskunov, and Kanel′ were extended to the case of a reaction diffusion equations with shear flow on an infinite cylinders $\Omega = \mathbb{R} \times \omega$, where ω is a bounded domain with smooth boundary,

$$u_t = \Delta u + \alpha(y) D_{x_1} u + f(u),$$

with $x = (x_1, y) \in \Omega$, for monostable, combustion and bistable nonlinearities with Neumann boundary conditions on $\partial\Omega$. For this case the travelling wave solutions have the form

$$u(x,t) = v(x_1 - ct, y),$$

where for example in the case of monostable and combustion type nonlinearities $v(\infty, y) = 1$ and $v(-\infty, y) = 0$ for all $y \in \omega$. Details of these results and their proofs can be found in the work of, Berestycki, Larrouturou and Lions [7], and, Berestycki and Nirenberg [10]. In this case when the ansatz is substituted into the reaction diffusion equation we get a second order elliptic equation on an infinite cylinder in terms of v

$$v_{\tau\tau} + \Delta_y v + (\alpha(y) - c)\, v_\tau + f(v).$$

The existence of travelling wave solutions was then shown by proving the existence of solutions to this elliptic equation on finite cylinders and then using the maximum principle to take the limit as the size of the cylinder tends to infinity. After the solution had been constructed, the sliding method, which is described in the paper by Berestycki and Nirenberg [9], was used to obtain uniqueness and monotonicity for some cases under additional conditions.

At a similar time work started to take place on the existence of travelling wave solutions in periodic media. In [53] and [54] Xin considered a reaction diffusion equation in $\mathbb{R}^n$ of the form

$$u_t = \operatorname{tr}\left(A(x)D^2u\right) + f(u),$$

where A is a symmetric positive definite matrix which is periodic in each component. For this equation Xin was able to show the existence of a travelling wave type solution,

$$u(x,t) = v\left(x \cdot k - ct, x\right),$$

where $k \in S^{n-1}$ and the profile function v is periodic in x. The methods used in these papers are in a similar vein to those used in the case of infinite cylinders in that existence is first proved on a finite cylinder then the limit as the size of cylinder tends to infinity is taken. However when we substitute this ansatz into the reaction diffusion equation we get a degenerate elliptic equation so an additional regularisation step is required.

Later in [50] Xin proved the existence of travelling wave solutions for the reaction diffusion convection equation

$$u_t = \operatorname{tr}\left(A(x)D^2u\right) + b(x) \cdot \nabla u + f(u),$$

with a combustion nonlinearity. Also in [29] Hudson and Zinner proved the existence of travelling wave solutions for a one dimensional reaction diffusion equation with a nonlinearity which was periodically dependent on the spatial variable

$$u_t = u_{xx} + f\left(x, u\right).$$

More recently the existence of travelling wave solutions in periodic media was dealt with in a very general setting for the reaction diffusion convection equation with combustion and monostable nonlinearities in the work of Berestycki and Hamel [4]. The key idea in this work is still to prove the existence on finite cylinders and then take the limit as the size of the cylinder tends to infinity. However because of the very general setting the details of how this is done are quite different to the previous work.

There have also been several approaches to proving the existence of travelling wave solutions using different methods. In [26] Heinze considered a reaction diffusion equation in a periodically perforated domain and was able to prove the existence of travelling wave solutions by perturbing away from the homogenised equation. Also in [38]

Matthies, Schneider and Uecker looked at the existence of a travelling wave solution for the reaction diffusion equation in an infinite cylinder $\Omega = \mathbb{R} \times [0, L]^d$ with periodic boundary conditions on the cross section

$$u_t = \Delta u + f\left(u, D_{x_1}u, \nabla_y u, y, \frac{x_1}{\varepsilon}\right),$$

where $(x_1, y) \in \Omega$. The method they used was to formulate the equation in terms of the profile function v as a spatial dynamical system and then use iterated normal forms and exponential averaging. For a similar equation Matthies and Wayne looked at the pinning of travelling waves in [39].

The idea of formulating the partial differential equation for the profile function as a spatial dynamical system, as was done in [38] and [39], is one of the key ideas we will use in our analysis. This idea originated in the work of Kirchgässner [33] and has since been used in many different situation to find solutions for partial differential equation on unbounded domains. A few examples related to our work are its use by Eckmann and Wayne [18] to find travelling front solution for the Swift-Hohenberg equation, Haragus and Schneider [25] to prove the existence of travelling wave fronts for the Taylor-Couette problem and, Schneider and Uecker [44] to show the existence of pulse solutions for Maxwell's equations. Further examples of this method and a detailed development of related techniques can be found in the book of Haragus and Iooss [24].

We note that methods of Heinze and, Matthies, Schneider and Uecker, have the advantage that they naturally extend to systems of equations which is not possible for methods which make use of the maximum principle, however their results are not as general.

In the background we have presented up to this point we have focused on the existence of travelling wave solutions as this is the question we will be most concerned with during this thesis. However there are several other questions associated with travelling wave solutions which we will not discuss in this thesis. Two such questions which where originally introduced in the work of Kolmogorov, Petrovsky and Piskunov [34] are determining the critical speed $c_\star$ for monostable nonlinearities and the stability of the travelling wave solutions. Investigation of both these question has since undergone a great deal of work, a small selection of the work on these problem are [5, 6, 22, 23, 27, 48] for determining the critical speed and [2, 8, 32, 36, 40, 42, 48, 53] for the stability of travelling wave solutions. Other examples of related problems are the pinning of travelling waves in periodic media, which as mentioned above was investigated in [39], and the behaviour of travelling wave solutions in the vanishing diffusion limit, for which the convergence of critical wave speed and wave profiles where investigated in [16, 17].

1.3 Overview and Results

In this section we will give an overview of the rest of this thesis with a short description of the results proved in each chapter.

We begin in chapter 2 by investigating the existence of generalised travelling wave solutions when the wave speed $c \neq 0$ is taken to be a fixed constant. We prove that if our nonlinearity is of the form,

$$f\left(\frac{x}{\varepsilon}, u\right) = \mu u + p\left(\frac{x}{\varepsilon}\right) q(u),$$

where $\mu \in \mathbb{R}$, $p \in H^2\left(T^2\right)$, the space of periodic Sobolev function with periodic cell $[0, 2\pi]^2$, and $q \in C^\infty(\mathbb{R})$ then for appropriate conditions on p and q we have two cases. Firstly when $\mu \neq 0$ is sufficiently close to zero there exists a generalised travelling wave solution

$$u(x,t) = v\left(x \cdot k - ct, \frac{x}{\varepsilon}\right),$$

which corresponds to a heteroclinic connection between equilibria. Thus

$$v(\tau, \xi) \to v^{\pm}(\xi) \text{ as } \tau \to \pm\infty,$$

where the equilibria $v^{\pm}(\xi)$ are stationary solutions of the reaction diffusion equation, thus $u(x) = v^{\pm}(x/\varepsilon)$ solves the static problem

$$0 = \operatorname{div}(A \nabla u) + \mu u + p\left(\frac{x}{\varepsilon}\right) q(u),$$

one of which is the zero solution.

On the other hand in the second case when μ is sufficiently close to certain critical values then we have the existence of a generalised travelling wave solution, but we do not know about their structure. However if we assume additional quite restrictive conditions on p then we can show that these solutions will also correspond to heteroclinic connections.

We saw in section 1.2 that the existence of travelling wave solutions that connect two stationary solutions of the reaction diffusion equation has been proved in a variety of different cases. In the cases of monostable and combustion type nonlinearities the travelling wave solution will connect 0 and 1. Alternatively in the case of a bistable nonlinearity as well as travelling waves which connect 0 and 1 there will exist travelling wave solutions which connect 0 and 1 with an intermediate stationary solution. The result we will prove in chapter 2 gives the existence of these types of solution when $\mu \neq 0$ is close to zero or close to certain values, provided p and q satisfy certain conditions. While these solutions do not in general connect between 0 and 1, they are connections between stationary solutions of the reaction diffusion equation. In particular in the case when μ is close to zero the generalised travelling wave solution will connect between 0 and a non-zero stationary solution.

The result in chapter 2 gives the existence of generalised travelling wave solutions for any wave speed $c \neq 0$ provided that μ is sufficiently close to zero or certain values. This condition is similar in nature to the condition that for a fixed monostable nonlinearity travelling wave solutions will exist provided $c \geqslant c_\star$ the critical speed. However rather than fixing the nonlinearity and varying the wave speed to find when solutions exist, we fix the wave speed and vary the parameter μ in the nonlinearity to determine for what values of μ solutions exist.

In chapter 3 we then move on to look at what happens if we do not take $c \neq 0$ to be a fixed constant and instead treat it as a parameter in the dynamical system we derive. Hence by allowing μ and c to vary close to zero we find that there are values of μ and c where different kinds of travelling wave solutions exist. Specifically this allows us to prove the existence of travelling wave solutions which have periodic and homoclinic structures.

In relation to previous work the results in chapter 3 relates to the existence of travelling wave solutions for the reaction diffusion equation if the wave speed is less than the critical wave speed. As in this case we deal with what happens to the travelling waves as the wave speed varies around 0.

After this in chapter 4 we consider what happens to the generalised travelling wave solutions which correspond to heteroclinic connections, that we proved the existence of in chapter 2, as $\varepsilon \to 0$. For these generalised travelling waves we are able to prove that for μ sufficiently close to zero these generalised travelling wave solutions can be approximated by a limiting profile v^0

$$v^\varepsilon \left(x \cdot k - ct, \frac{x}{\varepsilon} \right) = v^0 \left(x \cdot k - ct \right) + O(\varepsilon) \text{ uniformly in } x \cdot k - ct \text{ as } \varepsilon \to 0,$$

where v^0 solves a related ordinary differential equation. Thus we show that the generalised travelling wave solution which depends periodically on the spatial variable is of order ε close to a travelling wave profile which does not depend periodically on the spatial variable.

Finally in chapter 5 we conclude by discussing what possible future directions there are for this work.

Notation

Summation

Let $\{a_m^{\pm}\}_{m\in\mathbb{Z}^2} \subset \mathbb{C}$, then for convenience we will use the notation

$$\sum_{m\in\mathbb{Z}^2} a_m^{\pm} := \sum_{m\in\mathbb{Z}^2} a_m^{+} + \sum_{m\in\mathbb{Z}^2} a_m^{-}$$

and

$$\sum_{m\in\mathbb{Z}^2} \pm a_m^{\pm} := \sum_{m\in\mathbb{Z}^2} a_m^{+} - \sum_{m\in\mathbb{Z}^2} a_m^{-}.$$

Function Spaces

Let E and F be Banach spaces then we define the following spaces

$$C_b^k(E,F) := \left\{ f \in C^k(E,F) : \max_{0\leqslant l\leqslant k} \sup_{x\in E} \left\| D^j f(x) \right\| < \infty \right\}$$

the space of bounded k-times continuously differentiable function,

$$C_b^{0,1}(E,F) := \left\{ f \in C_b^0(E,F) : |f|_{\mathrm{lip}} := \sup_{x,y\in E, x\neq y} \frac{\|f(x)-f(y)\|}{\|x-y\|} < \infty \right\}$$

the space of bounded Lipschitz continuous functions and

$$C_\eta(\mathbb{R},E) := \left\{ f \in C^0(\mathbb{R},E) : \|f\|_\eta := \sup_{t\in\mathbb{R}} \mathrm{e}^{-\eta|t|} \|f(t)\| < \infty \right\}$$

the space of continuous function with exponentially bounded growth of rate η.

Also we define the periodic Sobolev spaces $H^s(T^2)$ to be the closure of the infinitely differentiable periodic functions on $[0,2\pi]^2$ in the Sobolev space $H^s\left([0,2\pi]^2\right)$.

Chapter 2

Travelling Waves for $\mu = \frac{1}{\varepsilon^2}\eta^T A\eta + \delta$

2.1 Problem and Result

Consider a reaction diffusion equation in $\mathbb{R}^2$ with a nonlinearity which is periodic in the spatial variable $x \in \mathbb{R}^2$

$$u_t = \operatorname{div}(A\nabla u) + f\left(\frac{x}{\varepsilon}, u\right), \tag{2.1}$$

where A is a real symmetric positive definite matrix, $\varepsilon > 0$ and f is the nonlinearity. We assume that the nonlinearity f is of the form

$$f\left(\frac{x}{\varepsilon}, u\right) = \mu u + p\left(\frac{x}{\varepsilon}\right) q(u); \tag{2.2}$$

where $\mu \in \mathbb{R}$, $p \in H^2(T^2)$, the space of periodic Sobolev functions on $[0, 2\pi]^2$, and $q \in C^2(\mathbb{R})$ with $q(0) = 0$, $q'(0) = 0$, and $q''(0) \neq 0$.

For this equation we will investigate the existence of generalised travelling wave solutions moving with speed $c \neq 0$ in a direction $k \in S^1_A = \{y \in \mathbb{R}^2 : y^T Ay = 1\}$, the unit circle with respect to the inner product $(x, y)_A = x^T Ay$ on $\mathbb{R}^2$, of the form

$$u(x, t) = v\left(x \cdot k - ct, \frac{x}{\varepsilon}\right); \tag{2.3}$$

where the profile function $v = v(\tau, \xi)$ is a function $v : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ which is periodic in ξ with periodic cell $[0, 2\pi]^2$, i.e.

$$v(\tau, \xi_1 + \xi_2) = v(\tau, \xi_1) \text{ for all } \xi_2 \in (2\pi\mathbb{Z})^2.$$

The approach we will take is to treat μ as a parameter and investigate for what value of μ generalised travelling wave solutions exist. This method leads to the following result.

Theorem 2.1.1. *Let $\varepsilon > 0$, $c \neq 0$, and $k \in S^1$ be fixed and*

$$\mu = \frac{1}{\varepsilon^2}\eta^T A\eta + \delta \tag{2.4}$$

for $\eta \in \mathbb{Z}^2$ and $\delta \in \mathbb{R}$ with $|\delta| > 0$ sufficiently small, then we have the following results:

1. *If $\eta = 0$ and $p \in H^2\left(T^2\right)$ is such that $\int_{T^2} p(s)ds \neq 0$, then there exists a generalised travelling wave solution corresponding to a heteroclinic connection between equilibria i.e.*
$$v(\tau, \xi) \to v^{\pm}(\xi) \text{ as } \tau \to \pm\infty.$$

2. *If $\chi(\eta) := \#\left\{m \in \mathbb{Z}^2 : m^T Am = \eta^T A\eta\right\} = 2$ and p is in a non-empty open subset of $H^2\left(T^2\right)$, then there exists a non-trivial generalised travelling wave solution. Furthermore with additional restrictions on p there is a non-empty open subset of $H^2\left(T^2\right)$, where this solution will be a heteroclinic connection between equilibria.*

Remark 2.1.2.

- For a more detailed version of the above theorem, where the conditions on p are given explicitly, see section 2.6.
- The equilibria $v^{\pm}(\xi)$ are stationary solution of the reaction diffusion equation, thus $u(x) = v^{\pm}(x/\varepsilon)$ solves
$$0 = \operatorname{div}(A\nabla u) + p\left(\frac{x}{\varepsilon}\right)q(u)$$
- The value of $\chi(\eta)$ determines the dimension of the centre manifold, in case (1) when $\eta = 0$ the dimension is $\chi(0) = 1$.
- By non-trivial solution we mean a solution that varies in τ.

The approach we take, to prove this theorem, is to find an equation for the profile function v and write it as a spatial dynamical system, using the idea of Kirchgässner [33]. We are then able to use a local centre manifold reduction to find small solutions of this spatial dynamical system.

2.2 Spatial Dynamical System Formulation

In this section we will derive a spatial dynamical system formulation for our problem. We start by substituting the ansatz

$$u(x,t) = v\left(x \cdot k - ct, \frac{x}{\varepsilon}\right), \tag{2.3}$$

where $v = v(\tau, \xi)$ is periodic in ξ with periodic cell $[0, 2\pi]^2$, into the reaction diffusion equation (2.1); to obtain an equation in terms of the profile function v

$$-cv_\tau = \frac{1}{\varepsilon^2}\mathrm{div}_\xi\left(A\nabla_\xi v\right) + \frac{2}{\varepsilon}k^T A\nabla_\xi v_\tau + v_{\tau\tau} + \mu v + f(\xi, v), \tag{2.5}$$

which is a second order elliptic partial differential equation for $(\tau, \xi) \in \mathbb{R} \times [0, 2\pi]^2$ with periodic boundary conditions on the cross section.

We formulate this equation as a spatial dynamical system by treating the unbounded direction τ as time and the $\delta \in \mathbb{R}$ which appears in μ as an extra dependent variable. Thus if we let $U = (v, v_\tau)$, then we obtain the spatial dynamical system

$$\begin{aligned} U_\tau &= \mathcal{A}U + F(U, \delta) \\ \delta_\tau &= 0, \end{aligned} \tag{2.6}$$

where

$$\mathcal{A} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{\varepsilon^2}\left(\mathrm{div}_\xi\left(A\nabla_\xi \cdot\right) + \eta^T A\eta\right) & -c - \frac{2}{\varepsilon}k^T A\nabla_\xi \end{bmatrix} \in \mathcal{L}(X, Z)$$

and

$$F(U, \delta) = F((u_1, u_2), \delta) = \begin{bmatrix} 0 \\ -\delta u_1 - pq(u_1) \end{bmatrix} \in C^2(X \times \mathbb{R}, X);$$

the spaces X and Z, of periodic functions, will be defined in the next section and we will prove that $\mathcal{A}$ and F have the stated properties.

2.3 Centre Manifold Reduction

In this section we will show that there exists a local centre manifold reduction for the spatial dynamical system introduced in the previous section.

For the spatial dynamical system (2.6) the existence of a local centre manifold reduction means that there exists a local manifold around $(0, 0) \in X \times \mathbb{R}$ on which we find solution to our dynamical system. The idea when constructing this manifold is to first split the phase space $X \times \mathbb{R}$ into two parts $X_c \times \mathbb{R}$, where the spectrum of $\mathcal{A}$ has zero real parts, and X_h where the spectrum of $\mathcal{A}$ has non-zero real part. Then once this is done we construct a map $\psi : X_c \times \mathbb{R} \to X_h$ and an open neighbourhood Ω of $(0, 0) \in X \times \mathbb{R}$ such that the local manifold we find solutions on is

$$\mathcal{M}_c := \{(U^c + \psi(U^c, \delta), \delta) : (U^c, \delta) \in X_c \times \mathbb{R}\} \cap \Omega.$$

For a more detailed introduction to the local centre manifold theorem where the necessary assumptions are given see Appendix A.

2.3.1 Result

We want to show the existence of a local centre manifold reduction, thus we want to prove the following result.

Proposition 2.3.1. *There exist a finite dimensional subspace $X_c \times \mathbb{R} \subset X \times \mathbb{R}$ and a projection π_c onto X_c. Letting $X_h = (Id - \pi_c)\, X$, there exists a neighbourhood of the origin $\Omega \subset X \times \mathbb{R}$ and a map $\psi \in C_b^2\,(X_c \times \mathbb{R}, X_h)$ with $\psi(0,0) = 0$ and $D\psi(0,0) = 0$, such that if $(U^c, \delta) : I \to X_c \times \mathbb{R}$ solves*

$$\begin{aligned} U^c_\tau &= \mathcal{A}U^c + \pi_c F\left(U^c + \psi\left(U^c, \delta\right), \delta\right), \\ \delta_\tau &= 0 \end{aligned}$$

for some interval $I \subset \mathbb{R}$, and $(U, \delta)(\tau) = (U^c(\tau) + \psi(U^c(\tau), \delta(\tau)), \delta(\tau)) \in \Omega$ for all $\tau \in I$ then (U, δ) solves

$$\begin{aligned} U_\tau &= \mathcal{A}U + F(U, \delta) \\ \delta_\tau &= 0. \end{aligned}$$

Remark 2.3.2. Solutions found using this proposition lie on the local centre manifold

$$\mathcal{M}_c := \{(U^c + \psi\left(U^c, \delta\right), \delta) : (U^c, \delta) \in X_c \times \mathbb{R}\} \cap \Omega.$$

Proof of Proposition 2.3.1

Proposition 2.3.1 is proved in two steps; first we show the existence of a local centre manifold reduction for the restricted system with $\delta = 0$

$$U_\tau = \mathcal{A}U + F(U, 0);$$

then we extend this result to the full system to verify the proposition.

Proof for Restricted System

For clarity we state the exact result that we will prove for the restricted system.

Proposition 2.3.3. *There exists a finite dimensional subspace $X_c \subset X$ with a projection π_c onto X_c. Letting $X_h = (Id - \pi_c)\, X$, there exists a neighbourhood of the origin $\Omega \subset X$ and a map $\psi \in C_b^2\,(X_c, X_h)$, with $\psi(0) = 0$ and $D\psi(0) = 0$, such that if $U^c : I \to X_c$ solves*

$$U^c_\tau = \mathcal{A}U_c + \pi_c F\left(U^c + \psi\left(U_c\right), 0\right),$$

for some interval $I \subset \mathbb{R}$, *and* $U(\tau) = U^c(\tau) + \psi(U^c(\tau)) \in \Omega$ *for all* $\tau \in I$ *then* U *solves*

$$U_\tau = \mathcal{A}U + F(U, 0).$$

The above result follows directly from the local centre manifold theorem A.0.3 found in Appendix A. Thus to prove this result we just need to check that the theorem is applicable; hence we check the hypotheses of the theorem (H1) - (H3) (which can be found in Appendix A).

However, before we can start checking these hypotheses, we first need to define the phase space Z of the problem. In order to do this we work out eigenvalues and eigenfunctions of $\mathcal{A}$; these will then be used to define the phase space. To workout these eigenvalues and eigenfunctions we consider $\mathcal{A}$ as a linear map from $H^2(T^2) \times H^1(T^2)$ into $H^1(T^2) \times L^2(T^2)$. Thus we investigate the eigenvalue equation

$$\mathcal{A}U = \lambda U \text{ for } \lambda \in \mathbb{C}$$

and we look for eigenfunctions U of the form

$$U = \begin{bmatrix} 1 \\ \lambda \end{bmatrix} \exp(\mathrm{i}m \cdot \xi) \text{ for } m \in \mathbb{Z}^2.$$

A function of this form is an eigenfunction if λ satisfies

$$\lambda^2 + \left(c + \frac{2\mathrm{i}}{\varepsilon}k^T Am\right)\lambda + \frac{1}{\varepsilon^2}\left(\eta^T A\eta - m^T Am\right) = 0. \tag{2.7}$$

Hence we find eigenvalues and eigenfunctions

$$\lambda_m^\pm := \frac{-\left(c + \frac{2\mathrm{i}}{\varepsilon}k^T Am\right) \pm \sqrt{\left(c + \frac{2\mathrm{i}}{\varepsilon}k^T Am\right)^2 + \frac{4}{\varepsilon^2}\left(m^T Am - \eta^T A\eta\right)}}{2}, \tag{2.8}$$

$$U_m^\pm := \begin{bmatrix} 1 \\ \lambda_m^\pm \end{bmatrix} \exp(\mathrm{i}m \cdot \xi) \text{ for } m \in \mathbb{Z}^2. \tag{2.9}$$

Now we can define the Hilbert space which will be the phase space for our analysis, for this definition and throughout this chapter we will use the summing conventions

$$\sum_{m\in\mathbb{Z}^2} a_m^\pm := \sum_{m\in\mathbb{Z}^2} a_m^+ + \sum_{m\in\mathbb{Z}^2} a_m^-$$

and

$$\sum_{m\in\mathbb{Z}^2} \pm a_m^\pm := \sum_{m\in\mathbb{Z}^2} a_m^+ - \sum_{m\in\mathbb{Z}^2} a_m^-,$$

$$Z := \left\{ U = \sum_{m\in\mathbb{Z}^2} \alpha_m^\pm U_m^\pm : \alpha_m^\pm \in \mathbb{C}, \overline{\alpha_m^\pm} = \alpha_{-m}^\pm \text{ and } \sum_{m\in\mathbb{Z}^2} \left|\alpha_m^\pm\right|^2 \left(1 + \left|\lambda_m^\pm\right|^4\right) < \infty \right\} \tag{2.10}$$

with the inner product

$$(U,V)_Z = \sum_{m\in\mathbb{Z}^2} \alpha_m^\pm \overline{\beta_m^\pm} \left(1 + \left|\lambda_m^\pm\right|^4\right)$$

for $U = \sum_{m\in\mathbb{Z}^2} \alpha_m^\pm U_m^\pm$ and $V = \sum_{m\in\mathbb{Z}^2} \beta_m^\pm U_m^\pm$. This is the space of Fourier series for the basis $\left\{U_m^\pm : m \in \mathbb{Z}^2\right\}$ with a weighted inner product.

Remark 2.3.4. From the definition of the space Z the set

$$S = \operatorname{Span}_{\mathbb{C}} \left\{U_m^\pm : m \in \mathbb{Z}^2\right\} \cap Z,$$

Z and the set of eigenfunction $\left\{U_m^\pm : m \in \mathbb{Z}^2\right\}$ will be a Hilbert basis for the complexification of Z which we denote by $Z_{\mathbb{C}} := Z \oplus \mathrm{i}Z$.

Lemma 2.3.5. *Z is a Hilbert space.*

Proof: Clearly Z is an inner product space so all we need to check is completeness. Thus let

$$U_n = \sum_{m\in\mathbb{Z}^2} \alpha_{m,n}^\pm U_m^\pm$$

be a Cauchy sequence in Z then

$$\sum_{m\in\mathbb{Z}^2} \left|\alpha_{m,n}^\pm - \alpha_{m,k}^\pm\right|^2 \left(1 + \left|\lambda_m^\pm\right|^4\right) \to 0 \text{ as } n,k \to \infty.$$

Thus

$$\sum_{m\in\mathbb{Z}^2} \alpha_{m,n}^\pm \sqrt{1 + \left|\lambda_m^\pm\right|^4}$$

is a Cauchy sequence in $\ell^2(\mathbb{C})$ and hence converges i.e.

$$\sum_{m\in\mathbb{Z}^2} \alpha_{m,n}^\pm \sqrt{1 + \left|\lambda_m^\pm\right|^4} \to \sum_{m\in\mathbb{Z}^2} \alpha_m^\pm \sqrt{1 + \left|\lambda_m^\pm\right|^4} \text{ in } \ell^2(\mathbb{C}) \text{ as } n \to \infty.$$

Thus it follows that $U_n \to U = \sum_{m\in\mathbb{Z}^2} \alpha_m^\pm U_m^\pm$ in Z as $n \to \infty$; hence Z is a Hilbert space.

□

Remark 2.3.6. Later we will show, in the proof of lemma 2.3.11, that there exists c and $C > 0$ such that

$$c(1 + |m|^{\frac{1}{2}}) \leqslant \left|\lambda_m^\pm\right| \leqslant C\left(1 + |m|\right),$$

for all $m \in \left\{n \in \mathbb{Z}^2 : \eta^T A\eta \neq n^T An\right\}$. Thus the growth of the eigenvalues $\lambda_m^{\pm}$ are bounded above and below in terms of m.

Now if we take $U = (u_1, u_2) \in Z$ and we rewrite each of its components in terms of the Fourier basis $\left\{\exp\left(\mathrm{i}m \cdot \xi\right) : m \in \mathbb{Z}^2\right\}$, for the periodic Sobolev spaces $H^s(T^2)$, then we can use the above growth rates to show that

$$Z \hookrightarrow H^1\left(T^2\right) \times H^{\frac{1}{2}}\left(T^2\right),$$

is a continuous embedding.

On the other hand if we have $V \in H^2\left(T^2\right) \times H^1\left(T^2\right)$ and we write V in terms of the eigenfunctions $U_m^{\pm}$ we see that

$$H^2\left(T^2\right) \times H^1\left(T^2\right) \hookrightarrow Z$$

is a continuous embedding.

Next we define the space X which $\mathcal{A}$ maps into Z. The natural choice for this space would be the domain of $\mathcal{A}$ with the graph norm, but for this to be a Banach space we need to check that $\mathcal{A}$ is a closed operator on Z.

Lemma 2.3.7. *$\mathcal{A}$ is a closed operator on Z.*

Proof: The idea of this proof is to show that $\mathcal{A}$ is a closed operator on $H^1\left(T^2\right) \times H^{\frac{1}{2}}\left(T^2\right)$ and then use the continuous embedding of Z into $H^1\left(T^2\right) \times H^{\frac{1}{2}}\left(T^2\right)$ to show closedness on Z.

To avoid confusion we denote the extended operator by $\hat{\mathcal{A}} : \mathrm{D}(\hat{\mathcal{A}}) \to H^1\left(T^2\right) \times H^{\frac{1}{2}}\left(T^2\right)$, where $\mathrm{D}(\hat{\mathcal{A}}) := \left\{U \in H^1\left(T^2\right) \times H^{\frac{1}{2}}\left(T^2\right) : \mathcal{A}U \in H^1\left(T^2\right) \times H^{\frac{1}{2}}\left(T^2\right)\right\}$.

Let $\hat{U}_n = (\hat{u}_n^1, \hat{u}_n^2) \in \mathrm{D}(\hat{\mathcal{A}})$ be a sequence such that $\hat{U}_n \to \hat{U} = (\hat{u}^1, \hat{u}^2)$ and $\hat{\mathcal{A}}\hat{U}_n = (\hat{v}_n^1, \hat{v}_n^2) \to \hat{V} = (\hat{v}^1, \hat{v}^2)$ in $H^1\left(T^2\right) \times H^{\frac{1}{2}}\left(T^2\right)$ as $n \to \infty$. Then

$$\mathcal{A}\hat{U}_n = \begin{pmatrix} \hat{u}_n^2 \\ -\frac{1}{\varepsilon^2}\left(\mathrm{div}_\xi\left(A\nabla_\xi \hat{u}_n^1\right) + (\eta^T A\eta)\hat{u}_n^1\right) - c\hat{u}_n^2 - \frac{2}{\varepsilon}k^T A\nabla_\xi \hat{u}_n^2 \end{pmatrix};$$

so $\hat{v}_n^1 = \hat{u}_n^2 \to \hat{u}^2$ as $n \to \infty$ in $H^1(T^2)$ and for $\phi \in C^\infty\left(T^2\right)$ we have

$$\begin{array}{ll} \langle \hat{v}_n^2, \phi\rangle = & \langle \hat{u}_n^1, -\frac{1}{\varepsilon^2}\left(\mathrm{div}_\xi\left(A\nabla_\xi\phi\right) + \left(\eta^T A\eta\right)\phi\right)\rangle + \langle \hat{u}_n^2, -c\phi + \frac{2}{\varepsilon}k^T A\nabla_\xi\phi\rangle \\ \downarrow & \qquad\qquad\qquad\qquad\qquad \downarrow \\ \langle \hat{v}^2, \phi\rangle & \langle \hat{u}^1, -\frac{1}{\varepsilon^2}\left(\mathrm{div}_\xi\left(A\nabla_\xi\phi\right) + \left(\eta^T A\eta\right)\phi\right)\rangle + \langle \hat{u}^2, -c\phi + \frac{2}{\varepsilon}k^T A\nabla_\xi\phi\rangle \end{array}$$

as $n \to \infty$ by Hölder's inequality. Hence

$$\hat{v}^2 = -\frac{1}{\varepsilon^2}\left(\mathrm{div}_\xi\left(A\nabla_\xi \hat{u}^1\right) + \left(\eta^T A\eta\right)\hat{u}^1\right) - c\hat{u}^2 - \frac{2}{\varepsilon}k^T A\nabla_\xi \hat{u}^2$$

in the sense of distributions and it follows that

$$\hat{\mathcal{A}}\hat{U} = \hat{V} \text{ and } \hat{U} \in \mathrm{D}(\hat{\mathcal{A}});$$

thus $\hat{\mathcal{A}}$ is a closed operator.

Now we want to show that $\mathcal{A}$ is a closed operator. Let $U_n = \left(u_n^1, u_n^2\right) \in \mathrm{D}(\mathcal{A}) := \{U \in Z : \mathcal{A}U \in Z\}$ be a sequence such that $U_n \to U = (u^1, u^2)$ and $\mathcal{A}U_n = (v_n^1, v_n^2) \to V = (v^1, v^2)$ in Z as $n \to \infty$. Then, by the continuous embedding of Z into $H^1\left(T^2\right) \times H^{\frac{1}{2}}\left(T^2\right)$, these convergences hold in $H^1\left(T^2\right) \times H^{\frac{1}{2}}\left(T^2\right)$ and since $\hat{\mathcal{A}}$ is an extension of $\mathcal{A}$ we have $\hat{\mathcal{A}}U = V$; which implies $\mathcal{A}U = V$ and $U \in \mathrm{D}(\mathcal{A})$. Hence $\mathcal{A}$ is a closed operator on Z.

□

Thus we take $X = \mathrm{D}(\mathcal{A})$ with the graph norm

$$\|U\|_X = \|U\|_Z + \|\mathcal{A}U\|_X .$$

In actual fact it is possible to characterise X in an alternative way which fits in nicely with how we defined Z;

$$X = \left\{ U = \sum_{m\in\mathbb{Z}^2} \alpha_m^{\pm} U_m^{\pm} : \alpha_m^{\pm} \in \mathbb{C}, \overline{\alpha_m^{\pm}} = \alpha_{-m}^{\pm} \text{ and } \sum_{m\in\mathbb{Z}^2} \left|\alpha_m^{\pm}\right|^2 \left(1 + \left|\lambda_m^{\pm}\right|^6\right) < \infty \right\} \tag{2.11}$$

with the inner product

$$(U,V)_X = \sum_{m\in\mathbb{Z}^2} \alpha_m^{\pm}\overline{\beta_m^{\pm}} \left(1 + \left|\lambda_m^{\pm}\right|^6\right)$$

for $U = \sum_{m\in\mathbb{Z}^2} \alpha_m^{\pm} U_m^{\pm}$ and $V = \sum_{m\in\mathbb{Z}^2} \beta_m^{\pm} U_m^{\pm}$.

Remark 2.3.8. Similarly to Remark 2.3.6 there exist continuous embeddings

$$H^3\left(T^2\right) \times H^2\left(T^2\right) \hookrightarrow X \hookrightarrow H^2\left(T^2\right) \times H^{\frac{3}{2}}\left(T^2\right).$$

We are now in a position to check the first hypothesis (H1).

Lemma 2.3.9. *$\mathcal{A} \in \mathcal{L}(X, Z)$ and $F \in C^2(X \times \mathbb{R}, X)$.*

Proof: Firstly for $U \in X$

$$\|\mathcal{A}U\|_Z \leqslant \|U\|_Z + \|\mathcal{A}U\|_Z = \|U\|_X ;$$

so $\mathcal{A} \in \mathcal{L}(X, Z)$. Now to show $F \in C^2(X \times \mathbb{R}, X)$ we first need to show that it makes sense as a map from $X \times \mathbb{R}$ into X; thus we want to show

$$F(U,\delta) = F((u_1,u_2),\delta) = \begin{pmatrix} 0 \\ -\delta u_1 - pq(u_1) \end{pmatrix} \in X$$

for all $(U,\delta) \in X \times \mathbb{R}$. Let $(U,\delta) \in X \times \mathbb{R}$ be arbitrary then, by the embeddings in remark 2.3.8, $U \in H^2(T^2) \times H^{\frac{3}{2}}(T^2)$ and therefore, since $u_1 \in H^2(T^2)$ and $q \in C^2(\mathbb{R})$, it follows from [46, Proposition 13.3.9] that $q(u_1) \in H^2(T^2)$. Thus, as $H^2(T^2)$ is an algebra, $p \in H^2(T^2)$ and $\delta \in \mathbb{R}$, we have that $-\delta u_1 - pq(u_1) \in H^2(T^2)$. Finally since $0 \in H^3(T^2)$ we get that

$$\begin{pmatrix} 0 \\ -\delta u_1 - pq(u_1) \end{pmatrix} \in H^3(T^2) \times H^2(T^2) \subset X.$$

Thus $F : X \times \mathbb{R} \to X$ is a well-defined map.

Now to prove the required regularity of F we can just differentiate. For $((h_1,h_2),h_3)$ and $((g_1,g_2),g_3) \in X \times \mathbb{R}$ we have

$$D_{(U,\delta)}F(U,\delta)(h_1,h_2,h_3) = \begin{pmatrix} 0 & 0 & 0 \\ -\delta - pq'(u_1) & 0 & 0 \\ 0 & 0 & -u_1 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}$$

and

$$D^2_{(U,\delta)}F(U,\delta)(h_1,h_2,h_3)(g_1,g_2,g_3) = \begin{pmatrix} 0 \\ -pq''(u_1)h_1g_1 \\ -g_1h_3 \end{pmatrix}.$$

Therefore, since the second derivative is continuous on $X \times \mathbb{R}$, we have the desired regularity.

□

Thus with the observation that $F(0,0) = 0$ and $D_{(U,\delta)}F(0,0) = 0$ we have checked hypothesis (H1) with $k = 2$. Next we will check hypothesis (H2); for this we need to prove some properties of the spectrum and define projections onto different parts of the space.

We start by proving a growth property for the real parts of the eigenvalues $\lambda_m^{\pm}$ which we calculated earlier with respect to m. This growth will allow us to prove that the spectrum of $\mathcal{A}$ has spectral gaps either side of the imaginary axis, consists of just the eigenvalues $\lambda_m^{\pm}$ for $m \in \mathbb{Z}^2$ and has only a finite number of eigenvalues on the imaginary axis.

Remark 2.3.10. As we are dealing with the spectrum of $\mathcal{A}$ we need to consider $\mathcal{A}$ acting on the complexification of the Banach space Z which we will denote by

$Z_{\mathbb{C}} := Z \oplus \mathrm{i}Z$.

Lemma 2.3.11. *$|Re\lambda_m^{\pm}| \to \infty$ as $|m| \to \infty$.*

Proof: For $k \in S_A^1$ there exists a $k_\perp \in S_A^1$ such that $\{k, k_\perp\}$ is a orthonormal basis for $\mathbb{R}^2$ with respect to the inner product $(x, y)_A = x^T Ay$. Thus for any $m \in \mathbb{Z}^2$ we have

$$m = (m, k)_A k + (m, k_\perp)_A k_\perp;$$

which implies that

$$m^T Am = (m, k)_A^2 + (m, k_\perp)_A^2 .$$

Now, using this decomposition of m and the expression for $m^T Am$ in the formula for $\lambda_m^{\pm}$ (2.8), we get that

$$\mathrm{Re}\lambda_m^{\pm} = \frac{1}{2}\left(-c \pm \mathrm{Re}\sqrt{c^2 + \frac{4}{\varepsilon^2}(m, k_\perp)_A^2 + \frac{4c\mathrm{i}}{\varepsilon}(m, k)_A - \frac{4}{\varepsilon^2}\eta^T A\eta}\right).$$

Hence, as

$$\begin{aligned}&\mathrm{Re}\sqrt{c^2 + \frac{4}{\varepsilon^2}(m, k_\perp)_A^2 + \frac{4c\mathrm{i}}{\varepsilon}(m, k)_A - \frac{4}{\varepsilon^2}\eta^T A\eta}\\ &= \frac{1}{\sqrt{2}}\sqrt{\left|c^2 + \frac{4}{\varepsilon^2}\left((m, k_\perp)_A^2 - \eta^T A\eta\right) + \frac{4c\mathrm{i}}{\varepsilon}(m, k)_A\right| + c^2 + \frac{4}{\varepsilon^2}\left((m, k_\perp)_A^2 - \eta^T A\eta\right)};\end{aligned}$$

which tends to ∞ as $|m| \to \infty$, it follows that $|\mathrm{Re}\lambda_m^{\pm}| \to \infty$ as $|m| \to \infty$.

□

Remark 2.3.12. The preceding lemma also gives upper and lower bounds on the growth of the real part of the eigenvalues with respect to m, since the growth will only depend on the growth of the square root. Thus there exist $c, C > 0$ such that for all $m \in \left\{n \in \mathbb{Z}^2 : \eta^T A\eta \neq n^T An\right\}$

$$c(1 + |m|^{\frac{1}{2}}) \leqslant \left|\mathrm{Re}\lambda_m^{\pm}\right| \leqslant C(1 + |m|). \tag{2.12}$$

Next we show that the spectrum has a gap either side of the imaginary axis by showing that the spectrum is equal to the set $\left\{\lambda_m^{\pm} : m \in \mathbb{Z}^2\right\}$ and then deducing that this set has gaps either side of the imaginary axis.

Lemma 2.3.13. *$\sigma(\mathcal{A}) = \left\{\lambda_m^{\pm} : m \in \mathbb{Z}^2\right\}$ and $\sigma(\mathcal{A})$ has spectral gaps either side of the imaginary axis.*

Proof: This result is proved by showing that any point not in $\left\{\lambda_m^{\pm} : m \in \mathbb{Z}^2\right\}$ is a resolvent point. Let $\lambda \in \mathbb{C} \setminus \left\{\lambda_m^{\pm} : m \in \mathbb{Z}^2\right\}$ be arbitrary then since $|\mathrm{Re}\lambda_m^{\pm}| \to \infty$ as

$|m| \to \infty$ we have that

$$\rho = \inf_{\mu \in \left\{\lambda_m^{\pm} : m \in \mathbb{Z}^2\right\}} |\lambda - \mu| > 0.$$

Now the set of eigenfunctions $\left\{U_m^{\pm} : m \in \mathbb{Z}^2\right\}$ is a Hilbert basis for $Z_{\mathbb{C}}$, so $\mathcal{S}_{\mathbb{C}} := \mathrm{Span}_{\mathbb{C}} \left\{U_m^{\pm} : m \in \mathbb{Z}\right\}$ is a dense subset of $Z_{\mathbb{C}}$. Hence, as $\mathcal{A}$ is a closed operator, to show that $(\mathcal{A} - \lambda I)$ has a bounded inverse on $Z_{\mathbb{C}}$ we just need to show that one exists on $\mathcal{S}_{\mathbb{C}}$.

Let $V = \sum_{m \in \mathbb{Z}^2} \alpha_m^{\pm} U_m^{\pm} \in \mathcal{S}_{\mathbb{C}}$; then we can define

$$U = \sum_{m \in \mathbb{Z}^2} \frac{\alpha_m^{\pm}}{\lambda_m^{\pm} - \lambda} U_m^{\pm} \in \mathcal{S}_{\mathbb{C}} \tag{2.13}$$

such that $(\mathcal{A} - \lambda I) U = V$ and

$$\|U\|_Z \leqslant \max_{\mu \in \sigma_p} \left\{ \frac{1}{|\mu - \lambda|} \right\} \|V\|_Z \leqslant \frac{1}{\rho} \|V\|_Z .$$

Thus we have a bounded inverse defined on $\mathcal{S}_{\mathbb{C}}$ and λ is a resolvent point. Hence $\sigma(\mathcal{A}) = \left\{\lambda_m^{\pm} : m \in \mathbb{Z}^2\right\}$ and furthermore, since $|\mathrm{Re}\lambda_m^{\pm}| \to \infty$ as $m \to \infty$, there can only be a finite number of eigenvalues near the imaginary axis. Thus there is a gap in the spectrum either side of the imaginary axis.

□

We have now shown that $\mathcal{A}$ has a spectral gap either side of the imaginary axis. The next step is to characterise the eigenvalues which lie on the imaginary axis.

Lemma 2.3.14. *An eigenvalue λ has zero real part if and only if $\lambda = \lambda_m^+$ for $m \in \mathbb{Z}$ such that*

$$m^T A m = \eta^T A \eta;$$

where η is the η chosen in theorem 2.1.1. Furthermore zero is the only eigenvalue with zero real part.

Proof: Suppose $\lambda = \lambda_m^+$ for $m^T A m = \eta^T A \eta$ then

$$\lambda_m^+ = \frac{-\left(c + \frac{2\mathrm{i}}{\varepsilon} k^T A m\right) + \sqrt{\left(c + \frac{2\mathrm{i}}{\varepsilon} k^T A m\right)^2}}{2} = 0$$

which has zero real part.

On the other hand suppose an eigenvalue λ has zero real part then λ solves

$$\lambda^2 + \left(c + \frac{2\mathrm{i}}{\varepsilon} k^T A l\right) \lambda + \frac{1}{\varepsilon^2} \left(\eta^T A \eta - l^T A l\right) = 0;$$

for some $l \in \mathbb{Z}^2$. Let $\tilde{\lambda}$ be the other root of this quadratic equation then by the properties of roots we have

$$\lambda\tilde{\lambda} = \frac{1}{\varepsilon^2}\left(\eta^T A\eta - l^T Al\right)$$
$$\lambda + \tilde{\lambda} = -\left(c + \frac{2\mathrm{i}}{\varepsilon}k^T Am\right).$$

Thus, since the real part of λ is zero, the first equations implies that either Re$\tilde{\lambda} = 0$ or Imλ= 0. However, as $c \neq 0$, the second equation tells use that Re$\tilde{\lambda} \neq 0$; hence Im$\lambda = 0$ and it follows that $\lambda = 0$. Then from rearranging the first equation we get $\eta^T A\eta = l^T Al$. Finally, since

$$\lambda_l^- = -\left(c + \frac{2\mathrm{i}}{\varepsilon}k^T Al\right) \neq 0$$

we have $\lambda = \lambda_l^+$.

Furthermore from these two calculations it follows that any eigenvalue with zero real part will be zero.

□

Remark 2.3.15. The proof, of the above lemma, also tells use that the number of eigenvectors associated with the zero eigenvalue will be

$$\chi(\eta) = \#\left\{m \in \mathbb{Z}^2 : \eta^T A\eta = m^T Am\right\}.$$

Thus the dimension of X_c will be $\chi(\eta)$.

The next step is to define the centre space X_c. Before doing this we introduce

$$\begin{aligned}
\mathcal{S} &= \left\{U_m^\pm | m \in \mathbb{Z}^2\right\}, & S &= \mathrm{span}_{\mathbb{C}}\left\{\mathcal{S}\right\} \cap Z,\\
\mathcal{S}^s &= \left\{U_m^\pm | \mathrm{Re}\lambda_m^\pm < 0\right\}, & S^s &= \mathrm{span}_{\mathbb{C}}\left\{\mathcal{S}^s\right\} \cap Z,\\
\mathcal{S}^u &= \left\{U_m^\pm | \mathrm{Re}\lambda_m^\pm > 0\right\}, & S^u &= \mathrm{span}_{\mathbb{C}}\left\{\mathcal{S}^u\right\} \cap Z,\\
\mathcal{S}^c &= \left\{U_m^\pm | \mathrm{Re}\lambda_m^\pm = 0\right\}, & S^c &= \mathrm{span}_{\mathbb{C}}\left\{\mathcal{S}^c\right\} \cap Z;
\end{aligned}$$

the last three lines correspond to the sets and spans of the stable, unstable and centre eigenfunctions. We define

$$X_c = Z_c := S^c; \tag{2.14}$$

the previous lemma tells us this is a finite dimensional space.

Now to finish checking hypothesis (H2) we need to define a projection $\pi_c \in \mathcal{L}(Z, X)$ onto X_c and check that it commutes with $\mathcal{A}$. π_c is defined in the following way: let $\tilde{\pi}_c$ be the projection of S onto S_c, defined in the natural way, then for each $U \in Z$ define

$\pi_c : Z \to X_c$ by

$$\pi_c U = \lim_{n\to\infty} \tilde{\pi}_c U_n;$$

where $U_n \in S$ is a sequence such that $U_n \to U$ in Z as $n \to \infty$.

Lemma 2.3.16. *π_c is well-defined and $\pi_c \in \mathcal{L}(Z, X)$.*

Proof: Let $U \in Z$ then, since S is a dense subset of Z, there exist a sequence $U_n \in S$ such that $U_n \to U$ in Z as $n \to \infty$. Now, since $\mathcal{A}\tilde{\pi}_c U_n = 0$ for all $n \in \mathbb{N}$, we have

$$\|\tilde{\pi}_c U_n - \tilde{\pi}_c U_m\|_X = \|\tilde{\pi}_c U_n - \tilde{\pi}_c U_m\|_Z$$

and it follows that $\tilde{\pi}_c U_n$ is a Cauchy sequence in X_c. Thus, as X_c is finite dimensional, $\tilde{\pi}_c U_n \to \pi_c U$ in X_c as $n \to \infty$. This limit is independent of the choice of sequence by an interlacing argument; hence π_c is well defined. Finally

$$\|\pi_c U\|_X = \|\pi_c U\|_Z \text{ for all } U \in Z;$$

so $\pi_c \in \mathcal{L}(Z, X)$.

□

Furthermore if we define Z_s, Z_u and Z_h to be the closures of S^s, S^u and $S^h := S^u \cup S^s$ in Z; then we can define projections π_s, π_u and $\pi_h \in \mathcal{L}(Z)$ and $\mathcal{L}(X)$ onto these spaces in a similar way. Hence we can decompose Z as follows

$$Z = Z_s \oplus X_c \oplus Z_u = X_c \oplus Z_h.$$

Finally, to complete the verification of (H2), it just remains to show that π_c commutes with $\mathcal{A}$.

Lemma 2.3.17. *π_c, π_s and π_u commute with $\mathcal{A}$.*

Proof: This result is proved in two steps; first we work out exactly how $\mathcal{A}$ acts on an element of X and then we use this to show that the projections commute with $\mathcal{A}$.

Let $U = \sum_{m\in\mathbb{Z}^2} \alpha_m^\pm U_m^\pm \in X$ then the partial sums $U_n = \sum_{|m|\leqslant n} \alpha_m^\pm U_m^\pm \to U$ in X as $n \to \infty$ and

$$\mathcal{A}U_n = \sum_{|m|\leqslant n} \lambda_m^\pm \alpha_m^\pm U_m^\pm \to \sum_{m\in\mathbb{Z}^2} \lambda_m^\pm \alpha_m^\pm U_m^\pm$$

in Z as $n \to \infty$. Therefore, as $\mathcal{A}$ is a closed operator, it follows that

$$\mathcal{A}U = \sum_{m\in\mathbb{Z}^2} \lambda_m^\pm \alpha_m^\pm U_m^\pm.$$

Now if we let $U = \sum_{m\in\mathbb{Z}^2} \alpha_m^\pm U_m^\pm \in X$ be arbitrary then we have that

$$\pi_c \mathcal{A} U = \pi_c \sum_{m\in\mathbb{Z}^2} \lambda_m^\pm \alpha_m^\pm U_m^\pm = \sum_{m\in\mathbb{Z}^2} \lambda_m^\pm \alpha_m^\pm \pi_c U_m^\pm = \mathcal{A}\pi_c U;$$

so $\mathcal{A}$ commutes with π_c. A similar argument works for π_s and π_u.

□

Hence we have completed the verification of hypothesis (H2) all that remains now is to check hypothesis (H3). To do this we first need to construct exponentially decaying semigroups on Z_s and Z_u.

Definition 2.3.18. *Let $T(t)$ be a C_0-semigroup on a Banach space E then we say T is an exponentially decaying semigroup if there exist γ and $C \geqslant 1$ such that*

$$\|T(t)\|_{\mathcal{L}(E)} \leqslant Ce^{-\gamma t}$$

for all $t \geqslant 0$. In this case we call γ the decay constant of the semigroup.

Lemma 2.3.19. *Let $X_s = D(\mathcal{A}) \cap Z_s$, $X_u = D(\mathcal{A}) \cap Z_u$, $\mathcal{A}_s = \mathcal{A}|_{X_s}$ and $\mathcal{A}_u = \mathcal{A}|_{X_u}$; then $\mathcal{A}_s$ and $-\mathcal{A}_u$ are closed operators which generate exponentially decaying C_0-semigroups of contractions on Z_s and Z_u respectively.*

Proof: We will prove this result for $\mathcal{A}_s$, an almost identical argument can be used for $-\mathcal{A}_u$.

We prove this result in two steps, first we show that $\mathcal{A}_s$ generates a C_0-semigroup of contractions; then we show that this semigroup is exponentially decaying.

To show that $\mathcal{A}_s$ generates a C_0-semigroup of contractions we use the Hille-Yoshida Theorem, which can be found in the book by Pazy [41, Theorem 1.3.1], thus we need to show

1. $\mathcal{A}_s$ is closed and $\overline{X_s} = Z_s$,
2. $\rho(\mathcal{A}_s) \supset \mathbb{R}^+$ and for every $\lambda > 0$ we have

$$\left\|(\mathcal{A}_s - \lambda I)^{-1}\right\| \leqslant \frac{1}{\lambda}.$$

Now for the first part $S^s \subset X_s$ so, as S^s is dense in Z^s, it follows that $\overline{X_s} = Z_s$. Furthermore the closedness of $\mathcal{A}_s$ follows from the closedness of $\mathcal{A}$; let $U_n \in X_s$ be a sequence such that $U_n \to U$ and $\mathcal{A}_s U_n \to V$ in Z as $n \to \infty$. Then, since $\mathcal{A}$ is closed, $\mathcal{A}U = V$ and $U \in \mathrm{D}(\mathcal{A})$ and, since $U_n \to U$ and Z_s is closed, we have that $U \in Z_s$. Hence $U \in X_s$ and $\mathcal{A}_s U = \mathcal{A}U = V$; so $\mathcal{A}_s$ is closed.

Now we prove the properties of the resolvent set for part 2. Let $\lambda \geqslant 0$ and $V = \sum_{m\in\mathbb{Z}^2} \alpha_m^{\pm} U_m^{\pm} \in S^s$, then if we take

$$U = \sum_{m\in\mathbb{Z}^2} \frac{\alpha_m^{\pm}}{\lambda_m^{\pm} - \lambda} U_m^{\pm} \in S^s;$$

we have that $(\mathcal{A} - \lambda I)U = V$ and

$$\|U\|_z \leqslant \frac{1}{\gamma + \lambda} \|V\|_Z , \tag{2.15}$$

where $\gamma = \min\{|\text{Re}\lambda_m^{\pm}| : \text{Re}\lambda_m^{\pm} \neq 0\} > 0$. Thus we have a bounded densely defined inverse and so, as $\mathcal{A}_s$ is a closed operator, we get that $\lambda \in \rho(\mathcal{A}_s)$ for all $\lambda \geqslant 0$ and we can extend this inverse and estimate to the whole of Z_s, to obtain our desired estimate for $\lambda > 0$. Hence by the Hille-Yoshida Theorem there exists a C_0-semigroup of contractions on Z_s, which we will denote by $T_s(\tau)$.

To complete the proof it just remains to show that $T_s(\tau)$ is an exponentially decaying semigroup. We prove this by using the spectrum of $T_s(\tau)$ to bound its norm.

The pairs $\left(\mathrm{e}^{\lambda_m^{\pm}\tau}, U_m^{\pm}\right)$ are an eigenpairs of $T_s(\tau)$ and, since the eigenfunctions $U_m^{\pm}$ form a Hilbert basis for the complexification of Z_s, we can use a similar argument to the one used to show that $\rho(\mathcal{A}) = \left\{\lambda_m^{\pm} : m \in \mathbb{Z}^2\right\}$ to deduce that these are all the whole spectrum of $T_s(\tau)$.

Now if we choose $U = \sum_{m\in\mathbb{Z}^2} \alpha_m^{\pm} U_m^{\pm} \in S^s$ arbitrarily then

$$\|T_s(\tau)U\|_Z \leqslant \mathrm{e}^{-\gamma\tau} \|U\|_Z$$

and this estimate can be extended to the whole of Z_s since S^s is dense in Z_s and $T_s(\tau)$ is a bounded linear operator. Thus $T_s(\tau)$ is an exponentially decaying semigroup.

□

Notation 2.3.20. The C_0-semigroups generated by $\mathcal{A}_s$ and $-\mathcal{A}_u$ will be denoted by $T_s(\tau)$ and $T_u(\tau)$, and their common decay constant will be denoted by

$$\gamma = \inf\left\{\left|\text{Re}\lambda_m^{\pm}\right| : \text{Re}\lambda_m^{\pm} \neq 0\right\}.$$

We are now in a position to prove hypothesis (H3) and confirm that we are able to perform a local centre manifold reduction.

Lemma 2.3.21. *Let $X_h := X_s \oplus X_u$ then for each $\eta \in [0, \gamma)$ and $f \in C_\eta(\mathbb{R}, X_h)$ the space of continuous functions with exponential growth of rate η, the affine problem*

$$U_\tau^h = \mathcal{A}U^h + f \textit{ and } U^h \in C_\eta(\mathbb{R}, X_h) \tag{2.16}$$

has a unique solution $U^h = K_h f$*, where* $K_h \in \mathcal{L}\left(C_\eta(\mathbb{R}, X_h)\right)$ *and*

$$\|K_h\|_{\mathcal{L}(C_\eta(\mathbb{R},X_h))} \leqslant \Gamma(\eta);$$

for some continuous function $\Gamma : [0, \gamma) \to \mathbb{R}^+$*.*

Proof: We construct a solution to the above affine problem by combining solutions to the problem when we restrict it to X_s and X_u.

Thus we start by considering the affine problem restricted to X_s, for $\eta \in [0, \gamma)$ and $f_s \in C_\eta(\mathbb{R}, X_s)$ find $U^s \in C_\eta(\mathbb{R}, X_s)$ such that

$$U^s_\tau = \mathcal{A}_s U^s + f_s. \tag{2.17}$$

Now as $\mathcal{A}_s$ generates an exponentially decaying C_0-semigroup of contractions we can define

$$U^s(\tau) = \int_{-\infty}^{\tau} T_s(\tau - \sigma) f_s(\sigma) \mathrm{d}\sigma,$$

we claim that this function solves the affine problem restricted to X_s (2.17).

Claim: $U^s \in C^1(\mathbb{R}, Z_s) \cap C_\eta(\mathbb{R}, X_s)$ and solves (2.17).

Proof of Claim: The first step is to check that U_s has the required regularity. We begin by proving $U^s \in C(\mathbb{R}, Z_s)$. Let $\tau \in \mathbb{R}$ be arbitrary and $h \in (-1, 1)$ then

$$\begin{aligned}
\|U^s(\tau + h) - U^s(\tau)\|_Z &= \left\| \int_{-\infty}^{\tau+h} T_s(\tau + h - \sigma) f_s(\sigma)\mathrm{d}\sigma - \int_{-\infty}^{\tau} T_s(\tau - \sigma) f_s(\sigma)\mathrm{d}\sigma \right\|_Z \\
&\leqslant \left\| \int_{-\infty}^{\tau} T_s(\tau - \sigma)\left(f_s(\sigma + h) - f_s(\sigma)\right)\mathrm{d}\sigma \right\|_Z \\
&\leqslant \left| \int_{-\infty}^{\tau} \|T_s(\tau - \sigma)\|_{\mathcal{L}(Z)} \|f_s(\sigma + h) - f_s(\sigma)\|_Z \right| \mathrm{d}\sigma.
\end{aligned}$$

Now, since $f_s \in C_\eta(\mathbb{R}, X_s)$ and T_s is exponentially decaying, for $k < \min\{0, \tau\}$ we have the estimate

$$\begin{aligned}
&\left| \int_{-\infty}^{k} \|T_s(\tau - \sigma)\|_{\mathcal{L}(Z)} \|f_s(\sigma + h) - f_s(\sigma)\mathrm{d}\sigma\|_Z \right| \\
&\qquad \leqslant \|f_s\|_\eta \int_{-\infty}^{k} \mathrm{e}^{-\gamma(\tau-\sigma)} \left(\mathrm{e}^{\eta|\sigma+h|} + \mathrm{e}^{\eta|\sigma|}\right) \mathrm{d}\sigma \\
&\qquad \leqslant 2\|f_s\|_\eta \mathrm{e}^{-\tau\gamma+\eta} \int_{-\infty}^{k} \mathrm{e}^{(\gamma-\eta)\sigma}\mathrm{d}\sigma \\
&\qquad \leqslant \frac{2}{\gamma - \eta} \|f_s\|_\eta \mathrm{e}^{-\tau\gamma+\eta} \mathrm{e}^{(\gamma-\eta)k} \to 0
\end{aligned}$$

as $k \to -\infty$. Thus if we let $\varepsilon > 0$ be arbitrary then there exists a $k < \min\{0, \tau\}$ such that

$$\left|\int_{-\infty}^{k} \|T_s(\tau-\sigma)\|_{\mathcal{L}(Z)} \|f_s(\sigma+h) - f_s(\sigma)\|_Z \, \mathrm{d}\sigma\right| < \frac{\varepsilon}{2}.$$

Now for this fixed k we know that f_s is uniformly continuous on $[k,\tau]$ so there exist a $\delta > 0$ such that if $|h| < \delta$

$$\left|\int_{k}^{\tau} \|T_s(\tau-\sigma)\|_{\mathcal{L}(Z)} \|f_s(\sigma+h) - f_s(\sigma)\|_Z \, \mathrm{d}\sigma\right| < \frac{\varepsilon}{2}.$$

Combining these two estimates we conclude that $\|U_s(\tau+h) - U_s(\tau)\| \to 0$ as $h \to 0$ and thus $U^s \in C(\mathbb{R}, Z_s)$.

The next step is to show that $U^s \in C(\mathbb{R}, X_s)$. In order to do this we first calculate $\mathcal{A}_s U^s$. Thus let $\tau \in \mathbb{R}$ be arbitrary then we can approximate $U^s(\tau)$ by the integrals

$$U_k^s(\tau) = \int_k^\tau T_s(\tau-\sigma) f_s(\sigma)\mathrm{d}\sigma,$$

for $k \leqslant \tau$ which will converge to $U^s(\tau)$ as $k \to -\infty$. Thus, since $\mathcal{A}_s$ is a closed operator,

$$\mathcal{A}_s U_k^s(\tau) = \int_k^\tau T_s(\tau-\sigma)\mathcal{A}_s f_s(\sigma)\mathrm{d}\sigma$$

and this integral converges to

$$\int_{-\infty}^\tau T_s(\tau-\sigma)\mathcal{A}_s f_s(\sigma)\mathrm{d}\sigma$$

as $k \to -\infty$. Therefore, as $\mathcal{A}$ is a closed operator, it follows that

$$\mathcal{A}_s U^s = \int_{-\infty}^\tau T_s(\tau-\sigma)\mathcal{A}_s f_s(\sigma)\mathrm{d}\sigma.$$

Now we are in a position to prove continuity in X_s. Let $h \in (-1,1)$ then, since X_s is equipped with the graph norm and $U^s \in C(\mathbb{R}, Z_s)$, all we need to prove is that $\|\mathcal{A}_s U^s(\tau+h) - \mathcal{A}_s U^s(\tau)\|_Z \to 0$ as $h \to 0$ for all $\tau \in \mathbb{R}$. Thus we let if $\tau \in \mathbb{R}$ be arbitrary then we have that,

$$\begin{aligned}
&\|\mathcal{A}_s U^s(\tau+h) - \mathcal{A}_s U^s(\tau)\| \\
&\qquad = \left\|\int_{-\infty}^{\tau+h} T_s(\tau+h-\sigma)\mathcal{A}_s f_s(\sigma)\mathrm{d}\sigma - \int_{-\infty}^{\tau} T_s(\tau-\sigma)\mathcal{A}_s f_s(\sigma)\mathrm{d}\sigma\right\|_Z \\
&\qquad \leqslant \left\|\int_{-\infty}^{\tau} T_s(\tau-\sigma)\left(\mathcal{A}_s f_s(\sigma+h) - \mathcal{A}_s f_s(\sigma)\right)\mathrm{d}\sigma\right\|_Z \\
&\qquad \leqslant \left|\int_{-\infty}^{\tau} \|T_s(t-\sigma)\|_{\mathcal{L}(Z)} \|\mathcal{A}_s f_s(\sigma+h) - \mathcal{A}_s f_s(\sigma)\|_Z \, \mathrm{d}\sigma\right|;
\end{aligned}$$

which tends to zero by a similar argument to the one given in the previous step since

$f_s \in C_\eta(\mathbb{R}, X_s)$.

Next we need to show that $U^s \in C^1(\mathbb{R}, Z_s)$ and satisfies (2.17). Let $\tau \in \mathbb{R}$ be arbitrary and $h > 0$ then

$$\begin{aligned}
\frac{U^s(\tau+h) - U^s(\tau)}{h} &= \frac{1}{h}\left(\int_{-\infty}^{\tau+h} T_s(\tau+h) f_s(\sigma)\mathrm{d}\sigma - \int_{-\infty}^{\tau} T_s(\tau) f_s(\sigma)\mathrm{d}\sigma\right) \\
&= \frac{T_s(h) - I}{h}\int_{-\infty}^{\tau} T_s(\tau) f_s(\sigma)\mathrm{d}\sigma + \frac{1}{h}\int_{\tau}^{\tau+h} T_s(\tau+h) f_s(\sigma)\mathrm{d}\sigma \\
&\to \mathcal{A}_s U^s(\tau) + f_s(\tau) \text{ as } h \searrow 0.
\end{aligned}$$

Thus the right derivative exists and is the required result. If we replace h with $-h$ then a similar argument will give the existence of the left derivative which is also the required result. Furthermore, since $\mathcal{A}_s U_s \in C(\mathbb{R}, Z_s)$ and $f_s \in C_\eta(\mathbb{R}, X_s)$, it follows that $U^s \in C^1(\mathbb{R}, Z_s)$ and satisfies (2.17).

Finally it just remains to check that $U^s \in C_\eta(\mathbb{R}, X_s)$, as $f_s \in C_\eta(\mathbb{R}, X_s)$ and T_s is an exponentially decaying semigroup, we have the estimate

$$\begin{aligned}
\|U^s(\tau)\|_X &= \|U^s(\tau)\|_Z + \|\mathcal{A}_s U^s(\tau)\|_Z \\
&= \left\|\int_{-\infty}^{\tau} T_s(\tau-\sigma) f_s(\sigma)\mathrm{d}\sigma\right\|_Z + \left\|\int_{-\infty}^{\tau} T_s(\tau-\sigma)\mathcal{A}_s f_s(\sigma)\mathrm{d}\sigma\right\|_Z \\
&\leqslant \left|\int_{-\infty}^{\tau} \|T_s(\tau-\sigma)\|_{\mathcal{L}(Z)} \left(\|f_s(\sigma)\|_Z + \|\mathcal{A}_s f_s(\sigma)\|_Z\right)\mathrm{d}\sigma\right| \\
&\leqslant \|f_s\|_\eta \int_{-\infty}^{\tau} \mathrm{e}^{-\gamma(\tau-\sigma)}\mathrm{e}^{\eta|\sigma|}\mathrm{d}\sigma \\
&= \|f_s\|_\eta \int_0^{\infty} \mathrm{e}^{-\gamma\sigma+\eta|\tau-\sigma|}\mathrm{d}\sigma \\
&\leqslant \frac{1}{\gamma-\eta}\|f_s\|_\eta \mathrm{e}^{\eta|\tau|};
\end{aligned}$$

hence $U^s \in C_\eta(\mathbb{R}, X_s)$ and we have completed the proof of the claim.
End of proof of Claim.

A similar argument to the one given above can be used for the affine problem on Z_u to show that for $f_u \in C_\eta(\mathbb{R}, X_u)$ the equation

$$U^u_\tau = -\mathcal{A}_u U^u + f_u$$

has a solution

$$U^u(\tau) = \int_{-\infty}^{\tau} T_u(\tau-\sigma) f_u(\sigma)\mathrm{d}\sigma \in C^1(\mathbb{R}, Z_u) \cap C_\eta(\mathbb{R}, X_u).$$

We will now use these solutions to construct a solution $U^h \in C_\eta(\mathbb{R}, X_h)$ for

$$U^h_\tau = \mathcal{A}U^h + f. \tag{2.16}$$

Since π_s and π_u commute with $\mathcal{A}$, we can split (2.16) into its stable and unstable parts

$$U^s_\tau + U^u_\tau = \mathcal{A}_s U^s + \mathcal{A}_u U^u + \pi_s f + \pi_u f;$$

where $U^s = \pi_s U$ and $U^u = \pi_u U$. Thus we can solve (2.16) by solving the equations on the stable and unstable parts separately. Hence we want to solve the pair of equations

$$U^s_\tau = \mathcal{A}_s U^s + \pi_s f \tag{2.18}$$
$$U^u_\tau = \mathcal{A}_u U^u + \pi_u f. \tag{2.19}$$

Now we have shown that (2.18) has a solution

$$U^s(\tau) = \int_{-\infty}^{\tau} T_s(\tau - \sigma)\pi_s f(\sigma)\mathrm{d}\sigma;$$

so we just need to construct a solution for (2.19). We consider the following auxiliary problem; suppose $U^u(\tau)$ solves (2.19) and let $V(\tau) = U^u(-\tau)$, then

$$\begin{aligned} V_\tau(\tau) = -U^u_\tau(-\tau) &= -\mathcal{A}_u U^u(-\tau) - \pi_u f(-\tau) \\ &= -\mathcal{A}_u V(\tau) - \pi_u f(-\tau). \end{aligned}$$

This equation for V has a solution

$$V(\tau) = -\int_{-\infty}^{\tau} T_u(\tau - \sigma)\pi_u f(-\sigma)\mathrm{d}\sigma;$$

which leads to a solution of (2.19)

$$U^u(\tau) = V(-\tau) = -\int_{\tau}^{\infty} T_u(\sigma - \tau)\pi_u f(\sigma)\mathrm{d}\sigma.$$

Now putting these solutions on the stable and unstable parts together we get a solution for (2.16),

$$U^h(\tau) = \int_{-\infty}^{\tau} T_s(\tau - \sigma)\pi_s f(\sigma)\mathrm{d}\sigma - \int_{\tau}^{\infty} T_u(\sigma - \tau)\pi_u f(\sigma)\mathrm{d}\sigma.$$

This solution is unique, since if we had two solutions then we could subtract them to get a solution of the homogeneous problem; however the homogeneous problem only has one solution in $C_\eta(\mathbb{R}, X_h)$ the zero solution. Hence the original two solutions must be equal.

Hence we define the map $K_h : C_\eta(\mathbb{R}, X_h) \to C_\eta(\mathbb{R}, X_h)$ by

$$K_h f(\tau) = \int_{-\infty}^{\tau} T_s(\tau - \sigma)\pi_s f(\sigma)\mathrm{d}\sigma - \int_{\tau}^{\infty} T_u(\sigma - \tau)\pi_u f(\sigma)\mathrm{d}\sigma. \tag{2.20}$$

K_h is a linear map since integration, T_s, T_u, π_s and π_s are all linear. Finally to complete the proof it just remains to estimate the norm of K_h. As $f \in C_\eta(\mathbb{R}, X_h)$ and since the semigroups are exponentially decaying we have the following estimate

$$\begin{aligned}
\|K_h f(\tau)\|_X &\leqslant \|\pi_s K_h f(\tau)\|_X + \|\pi_u K_h(\tau)\|_X \\
&\leqslant \left| \int_{-\infty}^{\tau} \|T_s(\tau-\sigma)\|_{\mathcal{L}(Z)} \|\pi_s f(\sigma)\|_X \,\mathrm{d}\sigma \right| \\
&\quad + \left| \int_{\tau}^{\infty} \|T_u(\sigma-\tau)\|_{\mathcal{L}(Z)} \|\pi_u f(\sigma)\|_X \,\mathrm{d}\sigma \right| \\
&\leqslant \|f\|_\eta \left(\int_{-\infty}^{\tau} \mathrm{e}^{-\gamma(\tau-\sigma)+\eta|\sigma|}\mathrm{d}\sigma + \int_{\tau}^{\infty} \mathrm{e}^{-\gamma(\sigma-\tau)+\eta|\sigma|}\mathrm{d}\sigma \right) \\
&\leqslant 2\|f\|_\eta \mathrm{e}^{\eta|t|} \int_0^{\infty} \mathrm{e}^{(\eta-\gamma)\sigma}\mathrm{d}\sigma \\
&\leqslant \frac{2}{\gamma-\eta}\|f\|_\eta \mathrm{e}^{\eta|t|}.
\end{aligned}$$

Hence it follows that

$$\|K_h f\|_\eta \leqslant \frac{2}{\gamma-\eta}\|f\|_\eta$$

and so finally we obtain

$$\|K_h\|_{\mathcal{L}(C_\eta(\mathbb{R},X_h))} \leqslant \frac{2}{\gamma-\eta}.$$

Thus we have completed the proof of the lemma.

□

Hence we have verified hypotheses (H1) - (H3); so we can perform a local centre manifold reduction. This completes the proof of Proposition 2.3.3 as it follows directly the local centre manifold theorem A.0.3 with the choices of X_c and π_c made in this section.

Proof for Full System

In the previous section we showed that there is a local centre manifold reduction for the restricted system. Now the aim of this section is to prove proposition 2.3.1 by extending the local centre manifold reduction for the restricted system to the whole system, which includes δ as a variable.

We start by writing the extended system in the form

$$\begin{pmatrix} U_\tau \\ \delta_\tau \end{pmatrix} = \mathcal{B}\begin{pmatrix} U \\ \delta \end{pmatrix} + H(U,\delta);$$

where

$$\mathcal{B} := \begin{pmatrix} \mathcal{A} & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{L}\left(X \times \mathbb{R}, Z \times \mathbb{R}\right)$$

and

$$H(U,\delta) := \begin{pmatrix} F\left(U,\delta\right) \\ 0 \end{pmatrix} \in C^2\left(X \times \mathbb{R}, X \times \mathbb{R}\right).$$

The properties stated above of $\mathcal{B}$ and H follow from lemma 2.3.9, since we are only extending $\mathcal{A}$ and F by zeros , and thus hypothesis (H1) holds for the extended system.

Furthermore the operator $\mathcal{B}$ will have the same eigenvalues as $\mathcal{A}$ with the eigenfunctions extended by 0 and an additional eigenfunction $(0,1)^T \in Z \times \mathbb{R}$ corresponding to the zero eigenvalue. Thus the centre space for the extended space is $X_c \times \mathbb{R}$ and the hyperbolic part will be unchanged from the restricted system. Now we can define a projection onto $X_c \times \mathbb{R}$ by $\tilde{\pi}_c(U,\delta) = (\pi_c U, \delta)$ for all $(U,\delta) \in Z \times \mathbb{R}$, where π_c is the projection for the restricted system.

Hence hypothesis (H2) holds for the extended system, since we have only extended π_c by the identity map on $\mathbb{R}$, so it will retain the required properties, and $\sigma(\mathcal{A}|_{X_c\times\mathbb{R}}) = 0$.

Finally, since the hyperbolic parts of the spaces are unchanged, lemma 2.3.21 shows that hypothesis (H3) holds.

Now to complete the proof of proposition 2.3.1 we apply the local centre manifold theorem A.0.3 with centre space $X_c \times \mathbb{R}$ and projection $\tilde{\pi}_c$. Doing this we get a map $\psi \in C_b^2\left(X_c \times \mathbb{R}, X_h\right)$ and an open set $\Omega \subset X \times \mathbb{R}$ such that if $(U^c, \delta) : \mathbb{R} \to X_c \times \mathbb{R}$ solves

$$\begin{pmatrix} U^c_\tau \\ \delta_\tau \end{pmatrix} = \mathcal{B}\begin{pmatrix} U^c \\ \delta \end{pmatrix} + \tilde{\pi}_c H\left(U^c + \psi\left(U^c, \delta\right), \delta\right),$$

for some interval $I \subset \mathbb{R}$, and $(U(\tau), \delta(\tau)) = \left(U^c(\tau) + \psi\left(U^c(\tau), \delta(\tau)\right), \delta(\tau)\right) \in \Omega$ for all $\tau \in I$ then $(U(\tau), \delta(\tau))$ solves

$$\begin{pmatrix} U_\tau \\ \delta_\tau \end{pmatrix} = \mathcal{B}\begin{pmatrix} U \\ \delta \end{pmatrix} + H(U,\delta);$$

Now writing this in terms of the X and $\mathbb{R}$ components we get that if $(U^c, \delta) : \mathbb{R} \to X_c \times \mathbb{R}$ solves

$$\begin{aligned} U^c_\tau &= \mathcal{A}U + \pi_c F\left(U^c + \psi\left(U^c, \delta\right), \delta\right), \\ \delta_\tau &= 0, \end{aligned}$$

for some interval $I \subset \mathbb{R}$, and $(U,\delta)(t) = (U^c(\tau) + \psi(U^c(\tau), \delta(\tau)), \delta(\tau)) \in \Omega$ for all $\tau \in I$

then (U, δ) solves

$$\begin{aligned} U_\tau &= \mathcal{A}U + F(U, \delta) \\ \delta_\tau &= 0. \end{aligned}$$

Hence we have proved proposition 2.3.1.

2.4 Solutions on the Centre Manifold

In the previous section we proved that small solutions for the infinite dimensional spatial dynamical system generated by our problem can be found by finding solutions on the finite dimensional local centre manifold. So, to complete the proof of theorem 2.1.1, we just need to find solutions on the local centre manifold which stay within the open neighbourhood Ω of the origin.

Notation 2.4.1. Through out this section we will need to use the expansions of the function p and q that appear in the nonlinearity,

$$F\left((u_1, u_2), \delta\right) = \begin{pmatrix} 0 \\ -\delta u_1 - pq(u_1) \end{pmatrix}.$$

Thus, since $p \in H^2\left(T^2\right)$ and $q \in C^2(\mathbb{R})$ with $q(0) = 0$, $q'(0) = 0$ and $q''(0) \neq 0$, we can write $p(\xi) = \sum_{m\in\mathbb{Z}^2} p_m \mathrm{e}^{im\cdot\xi}$ and $q(s) = q_2 s^2 + O(s^3)$.

2.4.1 Solutions on Centre Manifold for $\mu = \delta$

In this section we look at the case when $\chi(\eta) = 1$ and thus $\mu = \delta$. For this case, since the equation $m^T Am = 0$ only has one solution $(0, 0) \in \mathbb{Z}^2$, it follows from remark 2.3.15 that X_c is 1-dimensional and

$$X_c = \begin{cases} \mathrm{Span}_{\mathbb{R}}\left\{U_0^+\right\} & \text{if } c > 0 \\ \mathrm{Span}_{\mathbb{R}}\left\{U_0^-\right\} & \text{if } c < 0 \end{cases} = \mathrm{Span}_{\mathbb{R}}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}.$$

Throughout this section we will assume that $c > 0$ and $X_c = \mathrm{span}_{\mathbb{R}}\left\{U_c^+\right\}$. If $c < 0$ then the calculation from this section will work if roles of U_0^+ and U_0^- are interchanged.

We will be able to show that for $p \in H^2(T^2)$ with $\int_{T^2} p(s)\mathrm{d}s \neq 0$ and $\delta \in \mathbb{R}$ with $|\delta|$ sufficiently small, we get a solution on the local centre manifold given by a heteroclinic connection between equilibra.

By proposition 2.3.1, since $\sigma\left(\mathcal{A}|_{X_c}\right) = \{0\}$, we can find solutions on the local centre manifold by finding solutions to the equation

$$U^c_\tau = \pi_c F(U^c + \psi(U^c, \delta), \delta)$$
$$\delta_\tau = 0;$$

such that $(U^c(\tau) + \psi(U^c(\tau), \delta(\tau)), \delta(\tau)) \in \Omega$ for all $\tau \in \mathbb{R}$.

In order to find solutions to this equation we will rewrite the first equation as a real ordinary differential equation by letting $U^c(\tau) = y(\tau)U_0^+$, then we will use a blow-up rescaling to write this equation as an ordinary differential equation with terms up to quadratic order plus a small perturbation. Finally, since structurally stable solutions persist under small perturbation, we will find a solution for the original ordinary differential equation by finding a heteroclinic connection for the equation with only terms up to quadratic order.

To perform the rescaling and work out the equation with terms up to quadratic order, we first need to calculate the Taylor expansion terms of the reduction map ψ necessary to determine the equation on $X_c \times \mathbb{R}$ up to quadratic order in U^c.

Calculation of Reduction Map

In order to calculate the equation on X_c up to quadratic order in U^c we need to calculate all the linear terms and the quadratic term on the U_0^- component of the reduction map ψ. Thus from remark A.0.4 we need to satisfy

$$D_{(U_c,\delta)}\psi(U^c, \delta)\begin{pmatrix} \pi_c F(U^c + \psi(U^c, \delta), \delta) \\ 0 \end{pmatrix} = \mathcal{A}\psi(U^c, \delta) + \pi_h F(U^c + \psi(U^c, \delta), \delta) \tag{2.21}$$

for these terms. We do this by expanding the reduction map in terms of the eigenfunctions and solving the equation on each eigenfunction component for the relevant terms, thus we write

$$\psi(U^c, \delta) = \psi_0^-(U^c, \delta)\, U_0^- + \sum_{m\in\mathbb{Z}^2\setminus\{0\}} \psi_m^\pm(U^c, \delta)\, U_m^\pm.$$

Now if we let $\psi_0^+(U^c, \delta) := u_1^c$ and suppress the arguments of the functions $\psi_m^\pm$, then we can obtain equations for the projections of F onto X_c and X_h

$$\begin{aligned}\pi_c F(U^c + \psi(U^c, \delta), \delta) = &-\delta\left(\frac{\psi_0^+ + \psi_0^-}{\lambda_0^+ - \lambda_0^-}\right)U_0^+ \\ &-\left(\sum_{n,k\in\mathbb{Z}^2} q_2 p_n \frac{\left(\psi_k^+ + \psi_k^-\right)\left(\psi_{-n-k}^+ + \psi_{-n-k}^-\right)}{\lambda_0^+ - \lambda_0^-}\right)U_0^+\end{aligned}$$

and

$$\begin{aligned}\pi_h F\left(U^c + \psi\left(U^c,\delta\right),\delta\right) =& \delta\left(\frac{\psi_0^+ + \psi_0^-}{\lambda_0^+ - \lambda_0^-}\right)U_0^- \\ &+ \left(\sum_{n,k\in\mathbb{Z}^2} q_2 p_n \frac{\left(\psi_k^+ + \psi_k^-\right)\left(\psi_{-n-k}^+ + \psi_{-n-k}^-\right)}{\lambda_0^+ - \lambda_0^-}\right)U_0^- \\ &+ \sum_{m\in\mathbb{Z}^2\setminus\{0\}} \mp\left(\delta\left(\frac{\psi_m^+ + \psi_m^-}{\lambda_m^+ - \lambda_m^-}\right)U_m^\pm + \right. \\ &\left.\left(\sum_{n,k\in\mathbb{Z}^2} q_2 p_n \frac{\left(\psi_k^+ + \psi_k^-\right)\left(\psi_{m-n-k}^+ + \psi_{m-n-k}^-\right)}{\lambda_m^+ - \lambda_m^-}\right)U_m^\pm\right);\end{aligned}$$

up to quadratic order in U^c by substituting this expansion into F.

We now want to calculate the linear terms of the Taylor expansion of ψ. Thus let

$$\psi\left(U^c,\delta\right) = \left(L_0^-(\delta)U^c\right)U_0^- + \sum_{m\in\mathbb{Z}^2\setminus\{0\}}\left(L_m^\pm(\delta)U^c\right)U_m^\pm + O\left(\|U^c\|^2\right),$$

where the functions $L_m^\pm(\delta) : X_c \to \mathbb{C}$ are linear. Thus, if we substitute the equations for F on X_c and X_h into (2.21) and use the above expansion for ψ, we get that up to linear terms

$$\begin{aligned}&D_{(U^c,\delta)}\psi\left(U^c,\delta\right)\begin{pmatrix}\pi_c F\left(U^c + \psi\left(U^c,\delta\right),\delta\right)\\ 0\end{pmatrix} \\ &\quad= L_0^-(\delta)\left(-\delta\left(\frac{u_1^c + L_0^-(\delta)U^c}{\lambda_0^+ - \lambda_0^-}\right)U_0^+\right)U_0^- \\ &\quad\quad+ \sum_{m\in\mathbb{Z}\setminus\{0\}} L_m^\pm(\delta)\left(-\delta\left(\frac{u_1^c + L_0^-(\delta)U^c}{\lambda_0^+ - \lambda_0^-}\right)U_0^+\right)U_m^\pm + O\left(\|U^c\|^2\right)\end{aligned}$$

and

$$\begin{aligned}&\mathcal{A}\psi\left(U^c,\delta\right) + \pi_h F\left(U^c + \psi\left(U^c,\delta\right),\delta\right) \\ &\quad= \left(\lambda_0^- L_0^-(\delta)U^c\right)U_0^- + \sum_{m\in\mathbb{Z}^2\setminus\{0\}}\left(\lambda_m^\pm L_m^\pm(\delta)U^c\right)U_m^\pm \\ &\quad\quad+ \delta\left(\frac{u_1^c + L_0^-(\delta)U^c}{\lambda_0^+ - \lambda_0^-}\right)U_0^- + \delta\sum_{m\in\mathbb{Z}^2\setminus\{0\}}\mp\left(\frac{L_m^+(\delta)U^c + L_m^-(\delta)U^c}{\lambda_m^+ - \lambda_m^-}\right)U_m^\pm \\ &\quad\quad+ O\left(\|U^c\|^2\right).\end{aligned}$$

So on the U_0^- component of (2.21) we have the equation

$$-\delta L_0^-(\delta)\left(\left(\frac{u_1^c + L_0^-(\delta)U^c}{\lambda_0^+ - \lambda_0^-}\right)U_0^+\right) = \lambda_0^-\left(L_0^-(\delta)U^c\right) + \delta\left(\frac{u_1^c + L_0^-(\delta)U^c}{\lambda_0^+ - \lambda_0^-}\right). \quad (2.22)$$

Now, as $X_c = \mathrm{Span}_{\mathbb{R}}\left\{U_0^+\right\}$, we can write $U^c = yU_0^+$ for $y \in \mathbb{R}$ and $L_0^-(\delta)\left(yU_0^+\right) := a_0^- y$ for some $a_0^- \in \mathbb{R}$. Then, since $\lambda_0^+ = 0$ and $\lambda_0^- = -c$, the equation on the U_0^- component (2.22) becomes

$$-\delta a_0^-\left(\frac{y + a_0^- y}{c}\right) = -ca_0^- y + \delta\left(\frac{y + a_0^- y}{c}\right),$$

which can be rearranged to get

$$\delta\left(1 + a_0^-\right)^2 = c^2 a_0^-.$$

Thus we get two possible values for a_0^-

$$a_0^- = \frac{\left(c^2 - 2\delta\right) + \sqrt{\left(c^2 - 2\delta\right)^2 - 4\delta^2}}{2\delta} = \frac{c^2}{\delta} - 2 - \frac{2\delta}{c^2} + O(\delta^2) \text{ as } \delta \to 0$$

or

$$a_0^- = \frac{\left(c^2 - 2\delta\right) - \sqrt{\left(c^2 - 2\delta\right)^2 - 4\delta^2}}{2\delta} = \frac{2\delta}{c^2} + O(\delta^2) \text{ as } \delta \to 0;$$

however if we choose the first of these then the reduction map ψ would not satisfy the condition

$$D_{(U^c,\delta)}\psi(0,0) = 0.$$

So we take

$$a_0^- := \frac{\left(c^2 - 2\delta\right) - \sqrt{\left(c^2 - 2\delta\right)^2 - 4\delta^2}}{2\delta}.$$

Now on the $U_m^{\pm}$ components of (2.21) for $m \neq 0$ we have the equation

$$-\delta L_m^{\pm}(\delta)\left(\left(\frac{u_1^c + L_0^-(\delta)U^c}{\lambda_0^+ - \lambda_0^-}\right)U_0^+\right) = \lambda_m^{\pm}\left(L_m^{\pm}(\delta)U^c\right) \mp \delta\left(\frac{L_m^+(\delta)U_c + L_m^-(\delta)U^c}{\lambda_m^+ - \lambda_m^-}\right);$$

which is satisfied if we take $L_m^{\pm} = 0$. Hence we have determined the linear terms of the reduction map.

Next we want to find the quadratic terms on the U_0^- component. Thus we let

$$\psi\left(U^c, \delta\right) = \left(L_0^-(\delta)U^c + Q_0^-\left(U^c, \delta\right)\right)U_0^- + \sum_{m \in \mathbb{Z}^2 \setminus \{0\}} Q_m^{\pm}\left(U^c, \delta\right) + O\left(\|U^c\|^3\right);$$

where $Q_m^{\pm}(\cdot, \delta) : X_c \to \mathbb{C}$ are quadratic functions. Then, substituting the equations for F on X_c and X_h into (2.21) and using the above expansion for ψ in terms of the linear and quadratic terms, we get that the quadratic terms of

$$
\begin{aligned}
D_{(U^c,\delta)}\psi\left(U^c,\delta\right) & \begin{pmatrix} \pi_c F\left(U^c + \psi\left(U^c,\delta\right),\delta\right) \\ 0 \end{pmatrix} \\
&= L_0^-\left(-\delta\left(\frac{Q_0^-\left(U^c,\delta\right)}{\lambda_0^+ - \lambda_0^-}\right)U_0^+ - \frac{q_2 p_0}{\lambda_0^+ - \lambda_0^-}\left(u_1^c + L_0^-(\delta)U_c\right)^2 U_0^+\right)U_0^- \\
&\quad + D_{U^c}Q_0^-\left(U^c,\delta\right)\left(-\delta\left(\frac{u_1^c + L_0^-(\delta)U_c}{\lambda_0^+ - \lambda_0^-}\right)U_0^+\right)U_0^- \\
&\quad + \sum_{m\in\mathbb{Z}^2\setminus\{0\}} D_{U^c}Q_m^\pm\left(U^c,\delta\right)\left(-\delta\left(\frac{u_1^c + L_0^-(\delta)U_c}{\lambda_0^+ - \lambda_0^-}\right)U_0^+\right)U_m^\pm
\end{aligned}
$$

and the quadratic terms of

$$
\begin{aligned}
&\mathcal{A}\psi\left(U^c,\delta\right) + \pi_h F\left(U^c + \psi\left(U^c,\delta\right),\delta\right) \\
&\qquad = \lambda_0^- Q_0^-\left(U^c,\delta\right)U_0^- + \sum_{m\in\mathbb{Z}^2\setminus\{0\}} \lambda_m^\pm Q_m^\pm\left(U^c,\delta\right)U_m^\pm \\
&\qquad\quad + \delta\left(\frac{Q_0^-\left(U^c,\delta\right)}{\lambda_0^+ - \lambda_0^-}\right)U_0^- + \frac{q_2 p_0}{\lambda_0^+ - \lambda_0^-}\left(u_1^c + L_0^-(\delta)U_c\right)^2 U_0^- \\
&\qquad\quad + \sum_{m\in\mathbb{Z}^2\setminus\{0\}} \mp\left(\delta\left(\frac{Q_m^+\left(U^c,\delta\right) + Q_m^-\left(U^c,\delta\right)}{\lambda_m^+ - \lambda_m^-}\right)U_m^\pm + \right. \\
&\qquad\quad \left.\frac{q_2 p_m}{\lambda_m^+ - \lambda_m^-}\left(u_1^c + L_0^-(\delta)U^c\right)^2 U_m^\pm\right).
\end{aligned}
$$

Thus on the U_0^- component of (2.21) we have the equation

$$
\begin{aligned}
L_0^- & \left(-\delta\left(\frac{Q_0^-\left(U^c,\delta\right)}{\lambda_0^+ - \lambda_0^-}\right)U_0^+ - \frac{q_2 p_0}{\lambda_0^+ - \lambda_0^-}\left(u_1^c + L_0^-(\delta)U_c\right)^2 U_0^+\right) \\
&+ D_{U^c}Q_0^-\left(U^c,\delta\right)\left(-\delta\left(\frac{u_1^c + L_0^-(\delta)U_c}{\lambda_0^+ - \lambda_0^-}\right)U_0^+\right) \\
&= \lambda_0^- Q_0^-\left(U^c,\delta\right) + \delta\left(\frac{Q_0^-\left(U^c,\delta\right)}{\lambda_0^+ - \lambda_0^-}\right) + \frac{q_2 p_0}{\lambda_0^+ - \lambda_0^-}\left(u_1^c + L_0^-(\delta)U_c\right)^2
\end{aligned} \tag{2.23}
$$

and, since $X_c = \mathrm{Span}_{\mathbb{R}}\left\{U_0^+\right\}$, we can let $U^c = yU_0^+$, $L_0^-(\delta)\left(yU_0^+\right) = a_0^- y$ and $Q_0^-\left(yU_0^+,\delta\right) = b_0^- y^2$ for some $b_0^- \in \mathbb{R}$. Thus the equation on the U_0^- component (2.23) becomes

$$
\begin{aligned}
&- a_0^-\left(\frac{\delta b_0^- y^2 + q_2 p_0\left(y + a_0^- y\right)^2}{c}\right) - 2b_0^- y\left(\delta\frac{\left(y + a_0^- y\right)}{c}\right) \\
&\qquad = -cb_0^- y^2 + \delta\frac{b_0^- y^2}{c} + \frac{q_2 p_0}{c}\left(y + a_0^- y\right)^2;
\end{aligned}
$$

rearranging we get

$$\left(c^2 - 3\delta\left(1 + a_0^-\right)\right) b_0^- = q_2 p_0 \left(1 + a_0^-\right)^3.$$

Hence we have

$$b_0^- = \frac{q_2 p_0 \left(1 + a_0^-\right)^3}{c^2 - 3\delta\left(1 + a_0^-\right)}.$$

Thus we have determined the terms of the reduction map necessary to calculate the equation on the X_c up to quadratic order in U^c.

Solutions on the Centre Manifold

We are now in a position where we can write down the equation on X_c up to quadratic order and use this equation to find solutions on the local centre manifold. The equation on $X_c \times \mathbb{R}$ is

$$\begin{aligned} U^c_\tau &= \pi_c F(U^c + \psi(U^c, \delta), \delta) \\ \delta_\tau &= 0. \end{aligned}$$

Using the expansions of F and ψ from the last section, we can write this equation as

$$\begin{aligned} U^c_\tau = &-\frac{\delta}{\lambda_0^+ - \lambda_0^-}\left(u_1^c + L_0^-(\delta)U^c + Q_0^-\left(U^c, \delta\right)\right) U_0^+ \\ &\qquad - \frac{q_2 p_0}{\lambda_0^+ - \lambda_0^-}\left(u_1^c + L_0^- U^c\right)^2 U_0^+ + O\left(\|U^c\|^3\right) \\ \delta_\tau = &\ 0. \end{aligned}$$

Now if we let $U^c = yU_0^+$ then, since $\lambda_0 + = 0$, $\lambda_0^- = -c$, $L_0^-(\delta)\left(yU_0^-\right) = a_0^- y$ and $Q_0^-\left(yU^c, \delta\right) = b_0^- y^2$, we can write the equation on X_c in terms of y as

$$\begin{aligned} y_\tau &= -\frac{\delta}{c}\left(1 + a_0^-\right) y - \left(\frac{\delta b_0^- + q_2 p_0 \left(1 + a_0^-\right)^2}{c}\right) y^2 + O\left(y^3\right) \\ \delta_\tau &= 0. \end{aligned} \tag{2.24}$$

To find a solution of the above ordinary differential equation we perform a blow-up rescaling using the small parameter δ. Let

$$\tilde{y} = \frac{y}{\delta} \text{ and } \tilde{\tau} = \delta\tau$$

then

$$\tilde{y}_{\tilde{\tau}} = -\frac{\tilde{y} + q_2 p_0 \tilde{y}^2}{c} - \delta\left(\frac{a_0^- \tilde{y} + q_2 p_0 a_0^- \left(a_0^- + 2\right)\tilde{y}^2}{\delta c} + \frac{b_0^- \tilde{y}^2}{c} + O\left(\tilde{y}^3\right)\right)$$
$$\delta_{\tilde{\tau}} = 0. \tag{2.25}$$

and, since $a_0^- = O(\delta)$, we can find solutions to the above equation by viewing it as a perturbation of the equation

$$\tilde{y}_{\tilde{\tau}} = -\frac{\tilde{y} + q_2 p_0 \tilde{y}^2}{c}$$
$$\delta_{\tilde{\tau}} = 0, \tag{2.26}$$

for $|\delta|$ small.

In order to find our desired solution we will need the concept of structural stability. We say that a dynamical system is structurally stable in a open set U if any dynamical system which is sufficiently C^1-close to the dynamical system in U is topologically equivalent to the dynamical system. Furthermore we say a solution is structurally stable if there exists a open neighbourhood of the solution on which the dynamical system is structurally stable. Therefore if we can find a structurally stable solution for (2.26), then there will be a corresponding solution for the perturbed equation provided $|\delta|$ is sufficiently small.

Now $p_0 = \frac{1}{|T^2|}\int_{T^2} p(s)\mathrm{d}s \neq 0$ so equation (2.26) has two equilibria $\tilde{y} = 0$ and $\tilde{y} = -1/q_2 p_0$, with 0 stable and $-1/q_2 p_0$ unstable. Here by saying a equilibria is stable (unstable) we mean that the linearisation of the dynamical system at this equilibria is negative (positive). Hence there exists a heteroclinic connection between these two equilibria given by

$$\tilde{y}_1(\tau) = \begin{cases} \frac{-e^{-\frac{\tau}{c}}}{1+q_2 p_0 \mathrm{e}^{-\frac{\tau}{c}}} & \text{if } p_0 q_2 > 0, \\ \frac{e^{-\frac{\tau}{c}}}{1-q_2 p_0 \mathrm{e}^{-\frac{\tau}{c}}} & \text{if } p_0 q_2 < 0. \end{cases}$$

Thus, as heteroclinic connections between stable and unstable hyperbolic equilibria are structurally stable, there will be a heteroclinic connection between two equilibria for the perturbed equation (2.25) provided δ is sufficiently small. For details of why heteroclinic connection are structurally stable see Arnold [1] chapter 3.

If we denote the heteroclinic connection between the equilibria of the perturbed equation by $\tilde{y}_2(\tilde{\tau})$ then

$$\tilde{y}_2(\tilde{\tau}) \to \tilde{y}^{\pm} \text{ as } \tilde{\tau} \to \pm\infty.$$

Now inverting the blow-up rescaling we get a solution of the original equation (2.24)

$$y(\tau) = \delta\tilde{y}_2(\delta\tau)$$

such that

$$y(\tau) \to \delta\tilde{y}^{\pm} \text{ as } \tau \to \pm\infty$$

for a fixed $\delta \in \mathbb{R}$ with $|\delta|$ sufficiently small. This then corresponds to a solution on $X_c \times \mathbb{R}$ given by

$$(U^c(\tau), \delta(\tau)) = \left(y(\tau)U_0^+, \delta\right).$$

Finally to show that $(U^c + \psi(U^c, \delta), \delta)$ is a solution on the local centre manifold we need to show that $(U^c(\tau) + \psi(U^c(\tau), \delta), \delta) \in \Omega$, the open neighbourhood of the origin from proposition 2.3.1, for all $\tau \in \mathbb{R}$.

Since $y(\tau)$ scales with δ it follows that

$$\sup_{\tau\in\mathbb{R}} \|U^c(\tau)\|_X \to 0 \text{ as } \delta \to 0.$$

Then, as $\psi(0,0) = 0$ and ψ is continuous, it follows that for $|\delta|$ sufficiently small $(U^c(\tau) + \psi(U^c(\tau), \delta), \delta) \in \Omega$ for all $\tau \in \mathbb{R}$. So for $\delta \in \mathbb{R}$ with $|\delta|$ sufficiently small there exists a heteroclinic connection which is mapped on to the local centre manifold.

Thus by proposition 2.3.1 for δ sufficiently small we have a solution for the spatial dynamical system, which correspond to a heteroclinic connection between equilibria. Hence we have proved the first part of theorem 2.1.1.

2.4.2 Solutions on Centre Manifold for $\mu = \frac{1}{\varepsilon^2}\eta^T A\eta + \delta$

In this section we look at the case when $\chi(\eta) = 2$ and thus, $\mu = \frac{1}{\varepsilon^2}\eta^T A\eta + \delta$ (for $\eta \in \mathbb{Z}^2$) and the equation $m^T Am = \eta^T A\eta$ has only two solutions in $\mathbb{Z}^2$, given by $m = \pm\eta \in \mathbb{Z}^2$. Therefore lemma 2.3.14 tells us that

$$X_c = \mathrm{Span}_{\mathbb{C}}\left\{U_\eta^+, U_{-\eta}^+\right\} \cap Z = \left\{\alpha U_\eta^+ + \overline{\alpha} U_{-\eta}^+ : \alpha \in \mathbb{C}\right\}.$$

For this case we will show that there exist an open set of periodic functions $p \in H^2(T^2)$ for which there is a non-trivial global solution on the local centre manifold. Once we have this result we will show that if we make additional assumptions on p then this solution will be a heteroclinic connection between equilibria.

Again by proposition 2.3.1, since $\sigma(\mathcal{A}|_{X_c}) = \{0\}$, we can find global solutions on the local centre manifold by finding solutions to the equation

$$\begin{aligned} U_\tau^c &= \pi_c F(U^c + \psi(U^c, \delta), \delta) \\ \delta_\tau &= 0; \end{aligned}$$

such that $(U^c(\tau) + \psi(U^c(\tau), \delta(\tau)), \delta(\tau)) \in \Omega$ for all $\tau \in \mathbb{R}$.

To find solutions for this equation we will rewrite the first equation as a complex ordinary differential equation by letting $U^c(\tau) = z(\tau)U_\eta^+ + \overline{z}(\tau)U_{-\eta}^+$. Once we have done this we will then use a blow-up rescaling using the small parameter δ to write this equation as a ordinary differential equation with terms up to quadratic order plus a small perturbation. Finally we will rewrite the complex ordinary differential equation with terms up to quadratic order as a 2 dimensional real ordinary differential equation and use Conley index to find a non-trivial structurally stable solution. Then, since structurally stable solutions are preserved under small perturbations, this solution will correspond to a solution for the original equation for $\delta \in \mathbb{R}$ with $|\delta|$ sufficiently small.

As in the previous section before we can perform this argument, we need to work out the terms of the Taylor expansion of the reduction map necessary to find the equation on X_c up to quadratic order in U^c.

Calculation of Reduction Map

In order to calculate the equation on X_c up to quadratic order in U^c we will need to calculate all the linear terms and the quadratic terms on the U_η^+ and $U_{-\eta}^+$ components of the reduction map. Thus, by remark A.0.4, we need to satisfy the equation

$$D_{(U^c,\delta)}\psi\left(U^c,\delta\right)\begin{pmatrix}\pi_c F\left(U^c+\psi\left(U^c,\delta\right),\delta\right)\\ 0\end{pmatrix} = \mathcal{A}\psi\left(U^c,\delta\right)+\pi_h F\left(U^c+\psi\left(U^c,\delta\right),\delta\right); \tag{2.21}$$

for these terms. We do this by expanding the reduction map in terms of the eigenfunctions and solving the equation on each eigenfunction component. Thus we let

$$\psi\left(U^c,\delta\right) = \psi_\eta^-\left(U^c,\delta\right)U_\eta^- + \psi_{-\eta}^-\left(U^c,\delta\right)U_{-\eta}^- + \sum_{m\in\mathbb{Z}^2\setminus\{\pm\eta\}}\psi_m^\pm\left(U^c,\delta\right)U_m^\pm.$$

Now if we let $U_c = zU_\eta^+ + \overline{z}U_{-\eta}^+$ and, for notational convenience, set $\psi_\eta^+\left(U^c,\delta\right) := z$ and $\psi_{-\eta}^+\left(U^c,\delta\right) := \overline{z}$. Then, suppressing the arguments of the functions $\psi_m^\pm$ and substitute the above expansion into F, we get equations for the projections of F onto X_c and X_h,

$$\begin{aligned}\pi_c F\left(U^c+\psi\left(U^c,\delta\right),\delta\right) = &-\delta\left(\frac{\psi_\eta^+ + \psi_\eta^-}{\lambda_\eta^+ - \lambda_\eta^-}\right)U_\eta^+ - \delta\left(\frac{\psi_{-\eta}^+ + \psi_{-\eta}^-}{\lambda_{-\eta}^+ - \lambda_{-\eta}^-}\right)U_{-\eta}^+\\ &-\sum_{n,k\in\mathbb{Z}^2} q_2 p_{\eta+n}\frac{\left(\psi_k^+ + \psi_k^-\right)\left(\psi_{-n-k}^+ + \psi_{-n-k}^-\right)}{\lambda_\eta^+ - \lambda_\eta^-}U_\eta^+\\ &-\sum_{n,k\in\mathbb{Z}^2} q_2 p_{-\eta+n}\frac{\left(\psi_k^+ + \psi_k^-\right)\left(\psi_{-n-k}^+ + \psi_{-n-k}^-\right)}{\lambda_{-\eta}^+ - \lambda_{-\eta}^-}U_{-\eta}^+\end{aligned} \tag{2.27}$$

and

$$\begin{aligned}
\pi_h F\left(U^c + \psi\left(U^c, \delta\right), \delta\right) =& \delta\left(\frac{\psi_\eta^+ + \psi_\eta^-}{\lambda_\eta^+ - \lambda_\eta^-}\right) U_\eta^- + \delta\left(\frac{\psi_{-\eta}^+ + \psi_{-\eta}^-}{\lambda_{-\eta}^+ - \lambda_{-\eta}^-}\right) U_{-\eta}^- \\
&+ \sum_{n,k\in\mathbb{Z}^2} q_2 p_{\eta+n} \frac{\left(\psi_k^+ + \psi_k^-\right)\left(\psi_{-n-k}^+ + \psi_{-n-k}^-\right)}{\lambda_\eta^+ - \lambda_\eta^-} U_\eta^- \\
&+ \sum_{n,k\in\mathbb{Z}^2} q_2 p_{-\eta+n} \frac{\left(\psi_k^+ + \psi_k^-\right)\left(\psi_{-n-k}^+ + \psi_{-n-k}^-\right)}{\lambda_{-\eta}^+ - \lambda_{-\eta}^-} U_{-\eta}^- \\
&+ \sum_{m\in\mathbb{Z}^2\setminus\{\pm\eta\}} \mp\Bigg(\delta\left(\frac{\psi_m^+ + \psi_m^-}{\lambda_m^+ - \lambda_m^-}\right) U_m^\pm + \\
&\sum_{n,k\in\mathbb{Z}^2} q_2 p_{m+n} \frac{\left(\psi_k^+ + \psi_k^-\right)\left(\psi_{-k-n}^+ + \psi_{-k-n}^-\right)}{\lambda_m^+ - \lambda_m^-} U_m^\pm\Bigg) \qquad (2.28)
\end{aligned}$$

up to quadratic order in U^c.

Now we start finding the terms of the reduction map ψ by working out the its linear terms, so we let

$$\psi\left(U^c, \delta\right) = \left(L_\eta^-(\delta)U^c\right) U_\eta^- + \left(L_{-\eta}^-(\delta)U^c\right) U_{-\eta}^- + \sum_{m\in\mathbb{Z}^2\setminus\{\pm\eta\}} \left(L_m^\pm(\delta)U^c\right) U_m^\pm + O\left(\|U^c\|^2\right);$$

where the functions $L_m^\pm(\delta) : X_c \to \mathbb{C}$ are linear. Now if we let $U^c = zU_\eta^+ + \overline{z}U_{-\eta}^+$ for $z \in \mathbb{C}$. Then using the equations for the projections of F onto X_c and X_h with the above expansion some calculation shows that up to linear terms in U^c

$$\begin{aligned}
&D_{(U^c,\delta)}\psi\left(U^c, \delta\right)\begin{pmatrix} \pi_c F\left(U^c + \psi\left(U^c, \delta\right), \delta\right) \\ 0 \end{pmatrix} \\
&= L_\eta^-(\delta)\left(-\delta\left(\frac{z + L_\eta^-(\delta)U^c}{\lambda_\eta^+ - \lambda_\eta^-}\right) U_\eta^+ - \delta\left(\frac{\overline{z} + L_{-\eta}^-(\delta)U^c}{\lambda_{-\eta}^+ - \lambda_{-\eta}^-}\right) U_{-\eta}^+\right) U_\eta^- \\
&\quad + L_{-\eta}^-(\delta)\left(-\delta\left(\frac{z + L_\eta^-(\delta)U^c}{\lambda_\eta^+ - \lambda_\eta^-}\right) U_\eta^+ - \delta\left(\frac{\overline{z} + L_{-\eta}^-(\delta)U^c}{\lambda_{-\eta}^+ - \lambda_{-\eta}^-}\right) U_{-\eta}^+\right) U_{-\eta}^- \\
&\quad + \sum_{m\in\mathbb{Z}^2\setminus\{\pm\eta\}} L_m^\pm(\delta)\left(-\delta\left(\frac{z + L_\eta^-(\delta)U^c}{\lambda_\eta^+ - \lambda_\eta^-}\right) U_\eta^+ - \delta\left(\frac{\overline{z} + L_{-\eta}^-(\delta)U^c}{\lambda_{-\eta}^+ - \lambda_{-\eta}^-}\right) U_{-\eta}^+\right) U_m^\pm
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{A}\psi\left(U^c,\delta\right) + \pi_h F\left(U^c + \psi\left(U^c,\delta\right),\delta\right)\\
&\qquad = \left(\lambda_\eta^- L_\eta^-(\delta)U^c\right)U_\eta^- + \left(\lambda_{-\eta}^- L_{-\eta}^-(\delta)U^c\right)U_{-\eta}^- + \sum_{m\in\mathbb{Z}^2\setminus\{\pm\eta\}}\left(\lambda_m^\pm L_m^\pm(\delta)U^c\right)U_m^\pm\\
&\qquad\quad + \delta\left(\frac{z + L_\eta^-(\delta)U^c}{\lambda_\eta^+ - \lambda_\eta^-}\right)U_\eta^- + \delta\left(\frac{\overline{z} + L_{-\eta}^-(\delta)U^c}{\lambda_{-\eta}^+ - \lambda_{-\eta}^-}\right)U_{-\eta}^-\\
&\qquad\quad + \delta\sum_{m\in\mathbb{Z}^2\setminus\{\pm\eta\}}\mp\left(\frac{L_m^+(\delta)U^c + L_m^-(\delta)U^c}{\lambda_m^+ - \lambda_m^-}\right)U_m^\pm.
\end{aligned}$$

Thus on the U_η^- and $U_{-\eta}^-$ components of (2.21) we have the equations

$$\begin{aligned}
L_\eta^-(\delta)&\left(-\delta\left(\frac{z + L_\eta^-(\delta)U^c}{\lambda_\eta^+ - \lambda_\eta^-}\right)U_\eta^+ - \delta\left(\frac{\overline{z} + L_{-\eta}^-(\delta)U^c}{\lambda_{-\eta}^+ - \lambda_{-\eta}^-}\right)U_{-\eta}^+\right)\\
&= \lambda_\eta^- L_\eta^-(\delta)U^c + \delta\left(\frac{z + L_\eta^-(\delta)U^c}{\lambda_\eta^+ - \lambda_\eta^-}\right)
\end{aligned}$$

and

$$\begin{aligned}
L_{-\eta}^-(\delta)&\left(-\delta\left(\frac{z + L_\eta^-(\delta)U^c}{\lambda_\eta^+ - \lambda_\eta^-}\right)U_\eta^+ - \delta\left(\frac{\overline{z} + L_{-\eta}^-(\delta)U^c}{\lambda_{-\eta}^+ - \lambda_{-\eta}^-}\right)U_{-\eta}^+\right)\\
&= \lambda_{-\eta}^- L_{-\eta}^-(\delta)U^c + \delta\left(\frac{z + L_{-\eta}^-(\delta)U^c}{\lambda_\eta^+ - \lambda_\eta^-}\right).
\end{aligned}$$

However, since ψ maps into $X_h \subset X$, we have that $L_\eta^-(\delta) = \overline{L_{-\eta}^-(\delta)}$ and it follows that the two equation above are complex conjugates of each other.

Now, since $U^c = zU_\eta^+ + \overline{z}U_{-\eta}^+$, we have that $L_\eta^-(\delta)\left(zU_\eta^+ + \overline{z}U_{-\eta}^+\right) = \alpha_\eta^- z + \beta_\eta^- \overline{z}$ for some $\alpha_\eta^-, \beta_\eta^- \in \mathbb{C}$ and we can express the equation on U_η^- in terms of z to get,

$$\begin{aligned}
&-\delta\alpha_\eta^-\left(\frac{z + \left(\alpha_\eta^- z + \beta_\eta^-\overline{z}\right)}{\lambda_\eta^+ - \lambda_\eta^-}\right) - \delta\beta_\eta^-\left(\frac{\overline{z} + \left(\overline{\beta_\eta^-}z + \overline{\alpha_\eta^-}\overline{z}\right)}{\lambda_{-\eta}^+ - \lambda_{-\eta}^-}\right)\\
&\qquad = \lambda_\eta^-\left(\alpha_\eta^- z + \beta_\eta^-\overline{z}\right) + \delta\left(\frac{z + \left(\alpha_\eta^- z + \beta_\eta^-\overline{z}\right)}{\lambda_\eta^+ - \lambda_\eta^-}\right).
\end{aligned}$$

From this equation we can work out the coefficients α_η^- and β_η^- by equating the coefficients of the z and $\overline{z}$ terms,

$$z : -\delta \left(\frac{\alpha_\eta^- \left(1+\alpha_\eta^-\right)}{\lambda_\eta^+ - \lambda_\eta^-} + \frac{\left|\beta_\eta^-\right|^2}{\lambda_{-\eta}^+ - \lambda_{-\eta}^-} \right) = \lambda_\eta^- \alpha_\eta^- + \delta \left(\frac{1+\alpha_\eta^-}{\lambda_\eta^+ - \lambda_\eta^-} \right)$$

$$\overline{z} : -\delta \left(\frac{\alpha_\eta^- \beta_\eta^-}{\lambda_\eta^+ - \lambda_\eta^-} + \frac{\beta_\eta^- \left(1+\overline{\alpha_\eta^-}\right)}{\lambda_{-\eta}^+ - \lambda_{-\eta}^-} \right) = \lambda_\eta^- \beta_\eta^- + \delta \left(\frac{\beta_\eta^-}{\lambda_\eta^+ - \lambda_\eta^-} \right).$$

Now the $\overline{z}$ equation implies that $\beta_\eta^- = 0$, so rearranging the z equation we get the quadratic equation

$$\delta \left(\alpha_\eta^-\right)^2 + \left(2\delta + \lambda_\eta^- \left(\lambda_\eta^+ - \lambda_\eta^-\right)\right)\alpha_\eta^- + \delta = 0.$$

So we have two possible values for α_η^- namely,

$$\begin{aligned} \alpha_\eta^- &= \frac{-\left(2\delta + \lambda_\eta^- \left(\lambda_\eta^+ - \lambda_\eta^-\right)\right) + \sqrt{\left(2\delta + \lambda_\eta^- \left(\lambda_\eta^+ - \lambda_\eta^-\right)\right)^2 - 4\delta^2}}{2\delta} \\ &= O(\delta) \text{ as } \delta \to 0 \end{aligned}$$

or

$$\begin{aligned} \alpha_\eta^- &= \frac{-\left(2\delta + \lambda_\eta^- \left(\lambda_\eta^+ - \lambda_\eta^-\right)\right) - \sqrt{\left(2\delta + \lambda_\eta^- \left(\lambda_\eta^+ - \lambda_\eta^-\right)\right)^2 - 4\delta^2}}{2\delta} \\ &= -\frac{\lambda_\eta^- \left(\lambda_\eta^+ - \lambda_\eta^-\right)}{2\delta} - 2 + O(\delta) \text{ as } \delta \to 0. \end{aligned}$$

However if we choose the second of these values then the reduction map would not satisfy the condition,

$$D_{(U^c,\delta)}\psi(0,0) = 0.$$

Thus we choose the first of the values given above and hence we have determined $L_\eta^-(\delta)$ and $L_{-\eta}^-(\delta)$, since $L_{-\eta}^-(\delta)$ is the complex conjugate of $L_\eta^-(\delta)$.

Now on the $U_m^\pm$ component of equation (2.21) for $m \neq \pm\eta$ we have the equation,

$$\begin{aligned} -\delta L_m^\pm(\delta) &\left(\frac{z + L_\eta^-(\delta)U_c}{\lambda_\eta^+ - \lambda_\eta^-} U_\eta^+ + \frac{\overline{z} + L_{-\eta}^-(\delta)U^c}{\lambda_{-\eta}^+ - \lambda_{-\eta}^-} U_{-\eta}^+ \right) \\ &= \lambda_m^\pm L_m^\pm(\delta)U^c \mp \delta \left(\frac{L_m^+(\delta)U^c + L_m^-(\delta)U^c}{\lambda_m^+ - \lambda_m^-} \right); \end{aligned}$$

which is satisfied if we choose $L_m^\pm(\delta) = 0$ for all $m \neq \pm\eta$. Thus we have determined all the linear terms of the reduction map, we summaries this information below;

$$\begin{aligned}
L^-_\eta(\delta)\left(zU^+_\eta \overline{z}U^+_\eta\right) &= \alpha^-_\eta z\\
L^-_{-\eta}(\delta)\left(zU^+_\eta \overline{z}U^+_\eta\right) &= \overline{\alpha^-_\eta}\,\overline{z}\\
L^\pm_m(\delta)\left(zU^+_\eta \overline{z}U^+_\eta\right) &= 0 \text{ for all } m \neq \pm\eta.
\end{aligned}$$

Hence it just remains to determine the quadratic terms on the U^-_η and $U^-_{-\eta}$ components. Thus we let

$$\begin{aligned}
\psi\left(U^c,\delta\right) = &\left(L^-_\eta(\delta)U^c + Q^-_\eta\left(U^c,\delta\right)\right)U^-_{-\eta} + \left(L^-_{-\eta}(\delta)U^c + Q^-_{-\eta}\left(U^c,\delta\right)\right)U^-_{-\eta}\\
&+ \sum_{m\in\mathbb{Z}^2\setminus\{\pm\eta\}} Q^\pm_m\left(U^c,\delta\right)U^\pm_m + O\left(\|U^c\|^3\right);
\end{aligned}$$

where $Q^\pm_m(\cdot,\delta) : X_c \to \delta$ are quadratic functions. Then, if we let $U^c = zU^-_\eta + \overline{z}U^+_{-\eta}$ and use the equation for the projections of F onto X_c and X_h with the above expansion, we get that the quadratic terms in U^c of

$$\begin{aligned}
&D_{(U^c,\delta)}\psi\left(U^c,\delta\right)\begin{pmatrix}\pi_c F\left(U^c + \psi\left(U^c,\delta\right),\delta\right)\\ 0\end{pmatrix}\\
&\quad= \left(L^-_\eta(\delta)V\right)U^-_\eta + \left(L^-_{-\eta}(\delta)V\right)U^-_{-\eta} + \left(D_{U^c}Q^-_\eta\left(U^c,\delta\right)W\right)U^-_\eta\\
&\qquad+ \left(D_{U^c}Q^-_{-\eta}\left(U^c,\delta\right)W\right)U^-_{-\eta} + \sum_{m\in\mathbb{Z}^2\setminus\{\pm\eta\}}\left(D_{U^c}Q^\pm_m\left(U^c,\delta\right)W\right)U^\pm_m;
\end{aligned}$$

where

$$\begin{aligned}
V = &-\frac{\delta Q^-_\eta\left(U^c,\delta\right)}{\lambda^+_\eta - \lambda^-_\eta}U^+_\eta - \frac{\delta Q^-_{-\eta}\left(U^c,\delta\right)}{\lambda^+_{-\eta} - \lambda^-_{-\eta}}U^+_{-\eta}\\
&- \frac{q_2\left(p_{-\eta}\left(z + L^-_\eta(\delta)U^c\right)^2 + 2p_\eta\left|z + L^-_\eta(\delta)U^c\right|^2 + p_{3\eta}\left(\overline{z} + L^-_{-\eta}(\delta)U^c\right)^2\right)}{\lambda^+_\eta - \lambda^-_\eta}U^+_\eta\\
&- \frac{q_2\left(p_{-3\eta}\left(z + L^-_\eta(\delta)U^c\right)^2 + 2p_{-\eta}\left|z + L^-_\eta(\delta)U^c\right|^2 + p_\eta\left(\overline{z} + L^-_{-\eta}(\delta)U^c\right)^2\right)}{\lambda^+_{-\eta} - \lambda^-_{-\eta}}U^+_{-\eta}
\end{aligned}$$

and

$$W = -\delta\left(\frac{z + L^-_\eta U^c}{\lambda^+_\eta - \lambda^-_\eta}\right)U^+_\eta - \delta\left(\frac{\overline{z} + L^-_{-\eta}U^c}{\lambda^+_{-\eta} - \lambda_{-\eta}}\right)U^+_{-\eta},$$

and on the other hand using the same method we get that the quadratic terms in U^c

of

$$
\begin{aligned}
&\mathcal{A}\psi\left(U^c,\delta\right) + \pi_h F\left(U^c + \psi\left(U^c,\delta\right),\delta\right) \\
&\quad = \left(\lambda_\eta^- Q_\eta^-\left(U^c,\delta\right)\right)U_\eta^- + \left(\lambda_{-\eta}^- Q_{-\eta}^-\left(U^c,\delta\right)\right)U_{-\eta}^- + \sum_{m\in\mathbb{Z}^2\backslash\{\pm\eta\}} \left(\lambda_m^\pm Q_m^p m\left(U^c,\delta\right)\right)U_m^\pm \\
&\quad + \tilde{V} + \sum_{m\in\mathbb{Z}^2\backslash\{\pm\eta\}} \mp\left(\delta\left(\frac{Q_m^+\left(U^c,\delta\right) + Q_m^-\left(U^c,\delta\right)}{\lambda_m^+ - \lambda_m^-}\right)U_m^\pm\right. \\
&\quad \left. + q_2\left(\frac{p_{m-2\eta}\left(z + L_\eta^-(\delta)U^c\right)^2 + p_m\left|z + L_\eta^-(\delta)U^c\right|^2 + p_{m+2\eta}\left(\overline{z} + L_{-\eta}^-(\delta)U^c\right)^2}{\lambda_m^+ - \lambda_m^-}\right)U_m^\pm\right);
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{V} =& \frac{\delta Q_\eta^-\left(U^c,\delta\right)}{\lambda_\eta^+ - \lambda_\eta^-}U_\eta^- + \frac{\delta Q_{-\eta}^-\left(U^c,\delta\right)}{\lambda_{-\eta}^+ - \lambda_{-\eta}^-}U_{-\eta}^- \\
&+ \frac{q_2\left(p_{-\eta}\left(z + L_\eta^-(\delta)U^c\right)^2 + 2p_\eta\left|z + L_\eta^-(\delta)U^c\right|^2 + p_{3\eta}\left(\overline{z} + L_{-\eta}^-(\delta)U^c\right)^2\right)}{\lambda_\eta^+ - \lambda_\eta^-}U_\eta^- \\
&+ \frac{q_2\left(p_{-3\eta}\left(z + L_\eta^-(\delta)U^c\right)^2 + 2p_{-\eta}\left|z + L_\eta^-(\delta)U^c\right|^2 + p_\eta\left(\overline{z} + L_{-\eta}^-(\delta)U^c\right)^2\right)}{\lambda_{-\eta}^+ - \lambda_{-\eta}^-}U_{-\eta}^-.
\end{aligned}
$$

Now, as we only want to workout Q_η^- and $Q_{-\eta}^-$ we can just look at the equation on the U_η^- component, as $\overline{Q_\eta^-} = Q_{-\eta}^-$. The equation on the U_η^- component is

$$
\begin{aligned}
&L_\eta^-(\delta)V + D_{U^c}Q_\eta^-\left(U^c,\delta\right)W \\
&\qquad = \lambda_\eta^- Q_\eta^-\left(U^c,\delta\right) + \frac{\delta Q_\eta^-\left(U^c,\delta\right)}{\lambda_\eta^+ - \lambda_\eta^-} \\
&\qquad\quad + \frac{q_2\left(p_{-\eta}\left(z + L_\eta^-(\delta)U^c\right)^2 + 2p_\eta\left|z + L_\eta^-(\delta)U^c\right|^2 + p_{3\eta}\left(\overline{z} + L_{-\eta}^-(\delta)U^c\right)^2\right)}{\lambda_\eta^+ - \lambda_\eta^-}.
\end{aligned}
$$

Now we know that $L_\eta^-(\delta)\left(zU_\eta^+ + \overline{z}U_{-\eta}^+\right) = \alpha_\eta^- z$ and we can write $Q_\eta^-\left(zU_\eta^+ + \overline{z}U_{-\eta}^+\right) = \gamma_\eta^- z^2 + 2\zeta_\eta^-\left|z\right|^2 + \sigma_\eta^-\overline{z}^2$ for some $\gamma_\eta^-, \zeta_\eta^-, \sigma_\eta^- \in \mathbb{C}$. So therefore the above equation can be written in terms of z as

$$
\begin{aligned}
&-\frac{\alpha_\eta^-}{\lambda_\eta^+ - \lambda_\eta^-}\Bigg(\delta\left(\gamma_\eta^- z^2 + 2\zeta_\eta^- |z|^2 + \sigma_\eta^- \overline{z}^2\right) + q_2\Bigg(p_{-\eta}\left(1+\alpha_\eta^-\right)^2 z^2 + 2p_\eta \left|1+\alpha_\eta^-\right|^2 |z|^2 \\
&+ p_{3\eta}\left(1+\overline{\alpha_\eta^-}\right)^2 \overline{z}^2\Bigg)\Bigg) - 2\delta\left(\frac{\left(\gamma_\eta^- z + \zeta_\eta^- \overline{z}\right)\left(1+\alpha_\eta^-\right) z}{\lambda_\eta^+ - \lambda_\eta^-} + \frac{\left(\zeta_\eta^- z + \sigma_\eta^- \overline{z}\right)\left(1+\overline{\alpha_\eta^-}\right)\overline{z}}{\lambda_{-\eta}^+ - \lambda_{-\eta}^-}\right) \\
&\quad = \lambda_\eta^-\left(\gamma_\eta^- z^2 + 2\zeta_\eta^- |z|^2 + \sigma_\eta^- \overline{z}^2\right) + \frac{1}{\lambda_\eta^+ - \lambda_\eta^-}\Bigg(\delta\left(\gamma_\eta^- z^2 + 2\zeta_\eta^- |z|^2 + \sigma_\eta^- \overline{z}^2\right) \\
&\qquad + q_2\left(p_{-\eta}\left(1+\alpha_\eta^-\right)^2 z^2 + 2p_\eta\left|1+\alpha_\eta^-\right|^2 |z|^2 + p_{3\eta}\left(1+\overline{\alpha_\eta^-}\right)^2 \overline{z}^2\right)\Bigg),
\end{aligned}
$$

and if we now equate the coefficients of the z^2, $|z|^2$ and $\overline{z}^2$ terms and rearrange we get the following three equations.

$$
\begin{aligned}
z^2 : \left(\lambda_\eta^-\left(\lambda_\eta^+ - \lambda_\eta^-\right) + 3\delta\left(1+\alpha_\eta^-\right)\right)\gamma_\eta^- &= -q_2 p_{-\eta}\left(1+\alpha_\eta^-\right)^3 \\
|z|^2 : \left(\lambda_\eta^-\left(\lambda_\eta^+ - \lambda_\eta^-\right) + 2\delta\left(1+\alpha_\eta^-\right) + \delta\frac{\left(1+\overline{\alpha_\eta^-}\right)\left(\lambda_\eta^+ - \lambda_\eta^-\right)}{\lambda_{-\eta}^+ - \lambda_{-\eta}^-}\right)\zeta_\eta^- & \\
= -q_2 p_\eta\left(1+\alpha_\eta^-\right)^2\left(1+\overline{\alpha_\eta^-}\right) & \\
\overline{z}^2 : \left(\lambda_\eta^-\left(\lambda_\eta^+ - \lambda_\eta^-\right) + \delta\left(1+\alpha_\eta^-\right) + 2\delta\frac{\left(1+\overline{\alpha_\eta^-}\right)\left(\lambda_\eta^+ - \lambda_\eta^-\right)}{\lambda_{-\eta}^+ - \lambda_{-\eta}^-}\right)\sigma_\eta^- & \\
= -q_2 p_{3\eta}\left(1+\alpha_\eta^-\right)\left(1+\overline{\alpha_\eta^-}\right)^2 &
\end{aligned}
$$

These three equations determine γ_η^-, ζ_η^- and σ_η^- and thus we have determined $Q_\eta^-\left(U^c,\delta\right)$ and $Q_{-\eta}^-\left(U^c,\delta\right)$ to be

$$
\begin{aligned}
Q_\eta^-\left(zU_\eta^+ + \overline{z}U_{-\eta}^+,\delta\right) &= \gamma_\eta^- z^2 + 2\zeta_\eta^- |z|^2 + \sigma_\eta^- \overline{z}^2 \\
Q_{-\eta}^-\left(zU_\eta^+ + \overline{z}U_{-\eta}^+,\delta\right) &= \overline{\sigma_\eta^-} z^2 + 2\overline{\zeta_\eta^-} |z|^2 + \overline{\gamma_\eta^-}\,\overline{z}^2.
\end{aligned}
$$

Thus we have determined the terms of the reduction map needed to calculate the ordinary differential equation on $X_c \times \mathbb{R}$ up to quadratic order in U^c.

Equation on $X_c \times \mathbb{R}$

We are now in a position where we can write down the equation on X_c up to quadratic order in U^c and then use a blow-up rescaling to find the equation we need to analyse.

The equation on $X_c \times \mathbb{R}$ is

$$\begin{aligned} U^c_\tau &= \pi_c F\left(U^c + \psi\left(U^c, \delta\right), \delta\right) \\ \delta_\tau &= 0. \end{aligned}$$

Therefore, if we let $U^c(\tau) = z(\tau)U^+_\eta + \overline{z}(\tau)U^+_{-\eta}$ and use the equation for the projection of F onto X_c (2.27) together with expressions for the linear and quadratic terms from the last section, this equation becomes

$$\begin{aligned} &z_\tau U^+_\eta + \overline{z}_\tau U^+_{-\eta} \\ &\qquad = -\delta\left(\frac{z + \alpha^-_\eta z + \gamma^-_\eta z^2 + 2\zeta^-_\eta \left|z\right|^2 + \sigma^-_\eta \overline{z}^2}{\lambda^+_\eta - \lambda^-_\eta}\right) U^+_\eta \\ &\qquad\quad - \delta\left(\frac{\overline{z} + \overline{\alpha^-_\eta}\,\overline{z} + \overline{\sigma^-_\eta}\, z^2 + 2\overline{\zeta^-_\eta} \left|z\right|^2 + \overline{\gamma^-_\eta}\,\overline{z}^2}{\lambda^+_{-\eta} - \lambda^-_{-\eta}}\right) U^+_{-\eta} \\ &\qquad\quad - q_2\left(\frac{p_{-\eta}\left(1 + \alpha^-_\eta\right)^2 z^2 + 2p_\eta \left|1 + \alpha^-_\eta\right|^2 \left|z\right|^2 + p_{3\eta}\big(1 + \overline{\alpha^-_\eta}\big)^2 \overline{z}^2}{\lambda^+_\eta - \lambda^-_\eta}\right) U^+_\eta \\ &\qquad\quad - q_2\left(\frac{p_{-3\eta}\left(1 + \alpha^-_\eta\right)^2 z^2 + 2p_{-\eta} \left|1 + \alpha^-_\eta\right|^2 \left|z\right|^2 + p_\eta\big(1 + \overline{\alpha^-_\eta}\big)^2 \overline{z}^2}{\lambda^+_{-\eta} - \lambda^-_{-\eta}}\right) U^+_{-\eta} \\ &\qquad\quad + O\left(z^3\right) \\ &\delta_\tau = 0. \end{aligned}$$

Thus, as the equation on the $U^+_{-\eta}$ component is just the complex conjugate of the equation on the U^+_η component, we can solve the equation on $X_c \times \mathbb{R}$ by studying the system of equations

$$\begin{aligned} z_\tau = &- \frac{\delta\left(1 + \alpha^-_\eta\right) z + \left(q_2 p_{-\eta}\left(1 + \alpha^-_\eta\right)^2 + \delta\gamma^-_\eta\right) z^2 + 2\left(q_2 p_\eta \left|1 + \alpha^-_\eta\right|^2 + \delta\zeta^-_\eta\right) \left|z\right|^2}{\lambda^+_\eta - \lambda^-_\eta} \\ &- \frac{\left(q_2 p_{3\eta}\big(1 + \overline{\alpha^-_\eta}\big)^2 + \delta\sigma^-_\eta\right) \overline{z}^2}{\lambda^+_\eta - \lambda^-_\eta} + O\left(z^3\right) \\ \delta_\tau =& 0. \end{aligned} \tag{2.29}$$

The next step is to perform a blow-up rescaling using the small parameter δ to work out the equation that we will study. Thus if we let

$$\tilde{z} = \frac{z}{\delta} \text{ and } \tilde{\tau} = \delta\tau;$$

then the above equation becomes

$$\begin{aligned}
\tilde{z}_{\tilde{\tau}} =& -\frac{\tilde{z} + q_2\left(p_{-\eta}\tilde{z}^2 + 2p_\eta\left|\tilde{z}\right|^2 + p_{3\eta}\overline{\tilde{z}}^2\right)}{\lambda_\eta^+ - \lambda_\eta^-} \\
& - \delta\Bigg(\frac{\alpha_\eta^-\tilde{z} + q_2 p_{-\eta}\left(\left(1+\alpha_\eta^-\right)^2 - 1\right)\tilde{z}^2 + 2q_2 p_\eta\left(\left|1+\alpha_\eta^-\right|^2 - 1\right)\left|\tilde{z}\right|^2}{\delta\left(\lambda_\eta^+ - \lambda_\eta^-\right)} \\
& + \frac{q_2 p_{3\eta}\left(\left(1+\overline{\alpha_\eta^-}\right)^2 - 1\right)\overline{\tilde{z}}^2}{\delta\left(\lambda_\eta^+ - \lambda_\eta^-\right)} + \frac{\gamma_\eta^-\tilde{z}^2 + 2\zeta_\eta^-\left|z\right|^2 + \sigma_\eta^-\overline{\tilde{z}}^2}{\lambda_\eta^+ - \lambda_\eta^-} + O\left(\tilde{z}^3\right)\Bigg) \\
\delta_{\tilde{\tau}} =& 0.
\end{aligned} \tag{2.30}$$

Now, since $\alpha_\eta^- = O(\delta)$, we can find solutions to the above equation by viewing it as a perturbation of the equation

$$\begin{aligned}
\tilde{z}_{\tilde{\tau}} &= -\frac{\tilde{z} + q_2\left(p_{-\eta}\tilde{z}^2 + 2p_\eta\left|\tilde{z}\right|^2 + p_{3\eta}\overline{\tilde{z}}^2\right)}{\lambda_\eta^+ - \lambda_\eta^-} \\
\delta_{\tilde{\tau}} &= 0.
\end{aligned} \tag{2.31}$$

Thus if we can find a structurally stable solution for (2.31), then there will be a corresponding solution for the perturbed equation (2.30) provided δ is sufficiently small.

Now as a final step to obtain the required equation we write the complex ordinary differential equation (2.31) as a system of real ordinary differential equations. Hence if we let $\tilde{z}(\tilde{\tau}) = y_1(\tilde{\tau}) + \mathrm{i}y_2(\tilde{\tau})$ then (2.31) becomes,

$$\begin{aligned}
\frac{dy_1}{d\tilde{\tau}} + \mathrm{i}\frac{dy_2}{d\tilde{\tau}} &= -\frac{y_1 + \mathrm{i}y_2}{\lambda_\eta^+ - \lambda_\eta^-} - q_2\left(\frac{p_{-\eta}\left(y_1 + \mathrm{i}y_2\right)^2 + 2p_\eta\left(y_1^2 + y_2^2\right) + p_{3\eta}\left(y_1 - \mathrm{i}y_2\right)^2}{\lambda_\eta^+ - \lambda_\eta^-}\right) \\
\delta_{\tilde{\tau}} &= 0.
\end{aligned}$$

Now taking real and complex parts we turn the first equation into two real equation to get

$$\begin{aligned}
\frac{dy_1}{d\tilde{\tau}} =& -\mathrm{Re}\left\{\frac{1}{\lambda_\eta^+ - \lambda_\eta^-}\right\}y_1 + \mathrm{Im}\left\{\frac{1}{\lambda_\eta^+ - \lambda_\eta^-}\right\}y_2 - q_2\mathrm{Re}\left\{\frac{p_{-\eta} + 2p_\eta + p_{3\eta}}{\lambda_\eta^+ - \lambda_\eta^-}\right\}y_1^2 \\
& + 2q_2\mathrm{Im}\left\{\frac{p_{-\eta} - p_{3\eta}}{\lambda_\eta^+ - \lambda_\eta^-}\right\}y_1y_2 - q_2\mathrm{Re}\left\{\frac{2p_\eta - p_{-\eta} - p_{3\eta}}{\lambda_\eta^+ - \lambda_\eta^-}\right\}y_2^2 \\
\frac{dy_2}{d\tilde{\tau}} =& -\mathrm{Im}\left\{\frac{1}{\lambda_\eta^+ - \lambda_\eta^-}\right\}y_1 - \mathrm{Re}\left\{\frac{1}{\lambda_\eta^+ - \lambda_\eta^-}\right\}y_2 - q_2\mathrm{Im}\left\{\frac{2p_\eta - p_{-\eta} + p_{3\eta}}{\lambda_\eta^+ - \lambda_\eta^-}\right\}y_1^2 \\
& - 2q_2\mathrm{Re}\left\{\frac{p_{-\eta} - p_{3\eta}}{\lambda_\eta^+ - \lambda_\eta^-}\right\}y_1y_2 - q_2\mathrm{Im}\left\{\frac{2p_\eta - p_{-\eta} - p_{3\eta}}{\lambda_\eta^+ - \lambda_\eta^-}\right\}y_2^2 \\
\delta_{\tilde{\tau}} =& 0.
\end{aligned} \tag{2.32}$$

Thus we can find solutions to the equation on $X_c \times \mathbb{R}$ by finding solutions to the above equation which are structurally stable. In the next section we will study equations of this type using Conley Index.

Constructing Solutions via Conley Index

In this section we will consider the question of the existence of structurally stable solutions for ordinary differential equations of the type derived in the previous section. For convenience we will write these ordinary differential equation in the following ways

$$\begin{aligned}\frac{dy_1}{d\tau} &= \lambda_1 y_1 - \lambda_2 y_2 + \mathbf{y}^T M\mathbf{y} =: f_1(\mathbf{y})\\ \frac{dy_2}{d\tau} &= \lambda_2 y_1 + \lambda_1 y_2 + \mathbf{y}^T N\mathbf{y} =: f_2(\mathbf{y}),\end{aligned} \tag{2.33}$$

where

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \; N = \begin{pmatrix} n_1 & n_2 \\ n_2 & n_3 \end{pmatrix} \text{ and } M = \begin{pmatrix} m_1 & m_2 \\ m_2 & m_3 \end{pmatrix}.$$

Remark 2.4.2.

- We ignore the equation $\delta_\tau = 0$ for the moment, since this equation just tells us that δ is constant for solutions on the centre manifold.
- The coefficient λ_1 and λ_2 will correspond to the values given in equation (2.32), but the matrices M and N can be any symmetric matrices, since any possible matrices can be generated by choosing an appropriate periodic function p.

The tool we will use to prove the existence of structurally stable solutions for this ordinary differential equation is the Conley Index. A brief introduction to Conley index can be found in Appendix B.

Our strategy for proving the existence of non-trivial solutions will be the following: First we work out the Conley index of an isolating block which contains the equilibria of the equation. Once we have this Conley index we compare it to what the Conley index would be if there where no non-trivial solutions. Thus if these two indices are different, then there must be a non-trivial solution. However this method only works when we are able to work out the Conley index, as will be seen later in this section we are not always able to calculate the Conley index for our equation. But we can calculate the Conley index for a set of periodic functions $p \in H^2(T^2)$ which contains a non-empty open set.

Next we will explain our method for calculating the Conley index and give a particular example to illustrate this method.

The method we use to calculate the Conley index is as follows: Take a large rectangle which contains the equilibria of the ordinary differential equation. Now look at the sections of the boundary through which the vector field points in (entry set) and out (exit set). We then try to connect the entry sets to the exit set using solution curves. If we can do this we have constructed an isolating block and we can calculate the Conley index by quotienting by the exit set and taking the homotopy equivalence class (this is explained in more detail in Appendix B). We illustrate this process in the example below.

Example 2.4.3. Consider the ordinary differential equation

$$\frac{dy_1}{d\tau} = y_1 - y_1^2$$
$$\frac{dy_2}{d\tau} = y_2 - y_1.$$

This equation has two equilibria $(0, 0)$ and $(1, 1)$. We want to consider a rectangle which contains these equilibria, so let

$$R = \{(y_1, y_2) : -1 \leqslant y_1 \leqslant 3, -1 \leqslant y_2 \leqslant 2\}$$

and look at the direction of the vector field round the edge of this rectangle see figure 2-1.

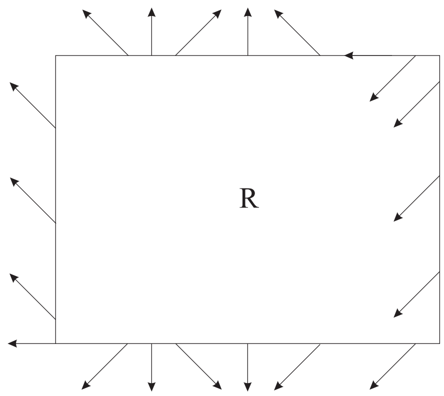


Figure 2-1: *Direction of vector field around edge of the rectangle R.*

Now to make this rectangle into a isolating block B we need to connect the entry sets to the exit sets using solution curves, the result of this process can be seen in figure 2-2.

The isolating block shown in figure 2-2 has an exit set which consists of only one subsection of the boundary, thus when we work out the Conley index we get

$$h(B) = \big[\big([b^+], [b^+]\big)\big] = \overline{0};$$

since we can retract $B/\sim$ to $[b^+]$.

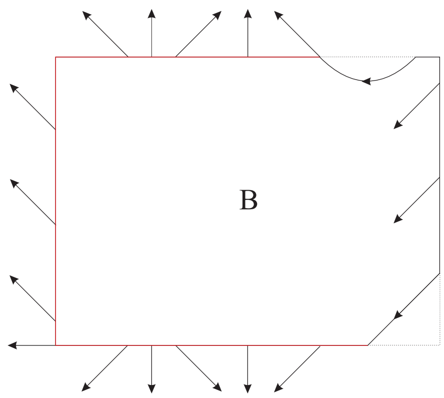


Figure 2-2: *The isolating block B constructed from the rectangle R, with the exit set b^+ marked in red.*

The above example illustrates how we can workout the Conley index for a particular ordinary differential equation. We will use the following series of steps to work out the Conley index of our general ordinary differential equation.

1. Take a large rectangle which contains the equilibria of the equation, then consider different cases depending on how many times the curves $f_1(\mathbf{y}) = 0$ and $f_2(\mathbf{y}) = 0$ intersect the boundary of the rectangle.

2. For each of these cases work out the possible arrangements of the points where the curves $f_1(\mathbf{y}) = 0$ and $f_2(\mathbf{y}) = 0$ intersect the boundary. Choosing the shape of the rectangles to reduce the number of cases as much as possible.

3. Now for each arrangement draw all the possible vector fields around the edge of the rectangle and turn these into isolating blocks when this is possible.

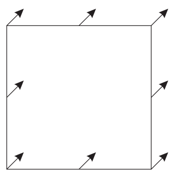
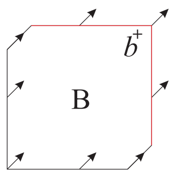
4. Workout the Conley index from these isolating blocks.

Following the above steps we consider the problem in three cases, when the curves $f_1(\mathbf{y}) = 0$ and $f_2(\mathbf{y}) = 0$ intersect the boundary of a sufficiently large rectangle 0, 4 and 8 times and $\det M, \det N \neq 0$. We deal with each of these cases in separate lemmata.

Remark 2.4.4. The sign of $\det M$ and $\det N$ determine what kind of conic section the curves $f_1(\mathbf{y}) = 0$ and $f_2(\mathbf{y}) = 0$ will be. If the determinant is greater than zero then the curve will be a circle or ellipse, on the other hand if the determinant is less than zero then the curve will be an hyperbola.

Lemma 2.4.5. *Let $\det N > 0$ and $\det M > 0$ then the Conley index of an isolating block which contains the equilibria of equation (2.33) will be equal to $[([b^+], [b^+])] =: \overline{0}$.*

Proof: Since $\det N > 0$ and $\det M > 0$ the curves $f_1(\mathbf{y}) = 0$ and $f_2(\mathbf{y}) = 0$ correspond to circles or ellipses. Thus it is possible to choose a sufficiently large rectangle, such that these curves will not intersect the boundary of the rectangle. This means that the vector field will never be parallel to the boundary of the rectangle. Thus we get a generic vector field shown in the table below, for which we can construct an isolating block and worked out the Conley index.

Direction of vector field round edge of rectangle.	Isolating block constructed from rectangle.	Conley index calculated from isolating block.
		$[([b^+], [b^+])] = \overline{0}$

Here and through out this section the exit set b^+ is marked in red.

Any other phase plane will be a rotation of this one and thus have the same Conley index.

□

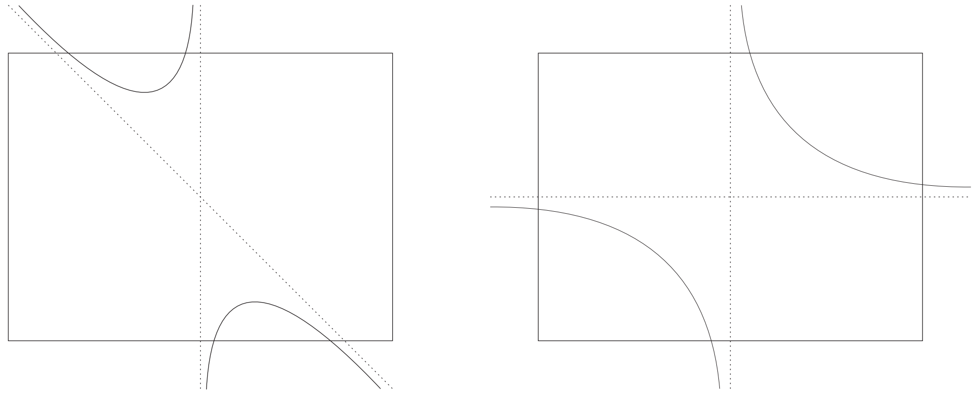
Next we turn our attention to the case when the curves $f_1(\mathbf{y}) = 0$ and $f_2(\mathbf{y}) = 0$ intersect the boundary of the rectangle 4 times.

Lemma 2.4.6. *Let* $\det M \cdot \det N < 0$, *then the Conley index of an isolating block containing the equilibria of equation* (2.33) *will be equal to* $[([b^+], [b^+])] = \overline{0}$.

Proof: Without loss of generality let $\det N < 0$ and $\det M > 0$, if this is not the case just interchange the roles of y_1 and y_2.

In this case the curve $f_1(\mathbf{y}) = 0$ corresponds to an ellipse or circle and again we can choose a rectangle large enough that this curve will not intersect its boundary. However the curve $f_2(\mathbf{y}) = 0$ will correspond to a hyperbola which will intersect the boundary at 4 points for a sufficiently large rectangle.

Thus we can split this case into two situations. As $|\mathbf{y}|$ gets large the hyperbola $f_2(\mathbf{y}) = 0$ will approach its asymptotes, if neither of these asymptotes are parallel to the y_1 axis then we can alway choose a rectangle such that the curve $f_2(\mathbf{y}) = 0$ intersects the top and bottom of the rectangle twice, we will call this situation 1 and it is illustrated in figure 2-3(a). On the other hand if one of the asymptotes is parallel to the y_1 axis then it is always possible to choose a rectangle such that the curve

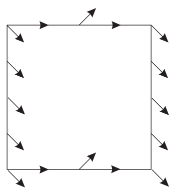
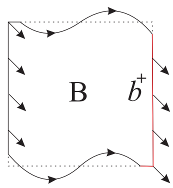
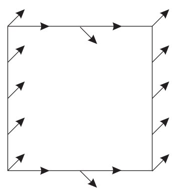
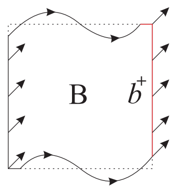


(a) Situation 1 (b) Situation 2

Figure 2-3: *Situations for lemma 2.4.6.*

$f_2(\mathbf{y}) = 0$ intersects each side once, we will call this situation 2 and it is illustrated in figure 2-3(b).

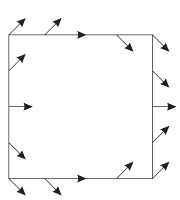
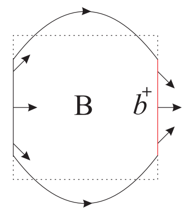
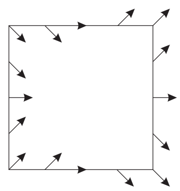
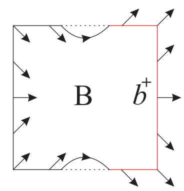
Now for situation 1 the vector field around the edge of the rectangle, up to reflection, will look like one of the cases given in the table below.

Direction of vector field round edge of rectangle.	Isolating block constructed from rectangle.	Conley index calculated from isolating block
		$[([b^+],[b^+])] = \overline{0}$
		$[([b^+],[b^+])] = \overline{0}$

Again here and through out this section the exit set b^+ is marked in red.

The other two possibilities with the vector field pointing to the left are just reflections of these two cases.

On the other hand for situation 2 the vector field, up to reflection will look like one of the cases given in the table below.

Direction of vector field round edge of rectangle.	Isolating block constructed from rectangle.	Conley index calculated from isolating block
		$[([b^+],[b^+])] = \overline{0}$
		$[([b^+],[b^+])] = \overline{0}$

Thus we have shown that for all vector field satisfying the conditions of the lemma the Conley index is $\overline{0}$.

□

We now move on to consider the final case when the curves $f_1(\mathbf{y}) = 0$ and $f_2(\mathbf{y}) = 0$ both intersect the boundary of the rectangle 4 times. This case is more involved and the method we have used in the previous two lemmata only works for some of the vector fields.

Lemma 2.4.7. *Let* $\det M < 0$ *and* $\det N < 0$*, then the Conley index of an isolating block containing the equilibria of equation* (2.33)*, when it can be calculated will be* $\overline{0}$*,* $[(S^1, [b^+])]$ *or* $[(S^1 \vee S^1, [b^+])]$*.*

Proof: In this case, since $\det M < 0$ and $\det N < 0$, the curves $f_1(\mathbf{y}) = 0$ and $f_2(\mathbf{y}) = 0$ both correspond to hyperbola. So, in the same way as we did in the proof of the previous lemma, we can always choose rectangles such that the curve $f_2(\mathbf{y}) = 0$ intersects the top and bottom sides twice (situation 1) or intersects each side once (situation 2). Thus we only need to worry about where the curve $f_1(\mathbf{y}) = 0$ intersects the rectangle.

Situation 1

In this situation the curve $f_2(\mathbf{y}) = 0$ intersects the boundary of the rectangle twice on the top and bottom sides. Thus if we consider the different possible cases when the asymptotes of $f_2(\mathbf{y}) = 0$ are parallel to or not parallel to the asymptotes of $f_1(\mathbf{y}) = 0$. We conclude that if we choose our rectangle large enough and of the correct shape we get the case show in figure 2-4.

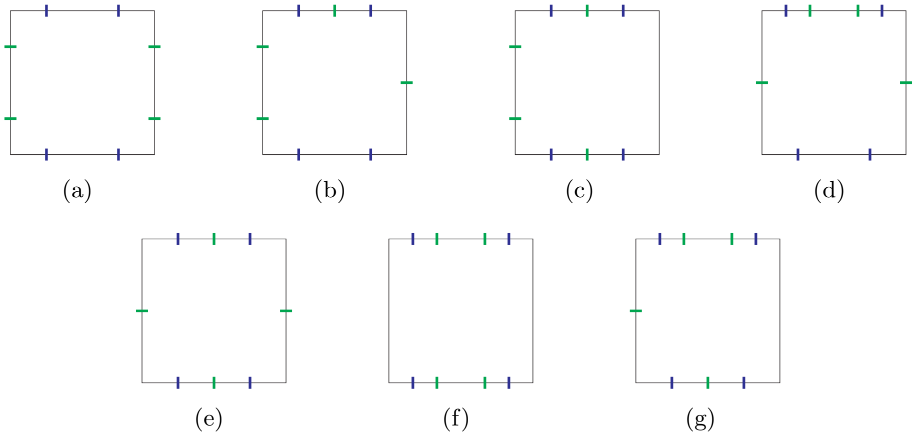
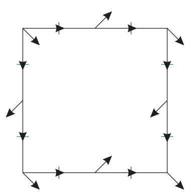
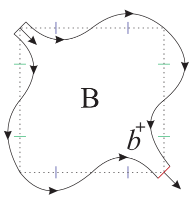
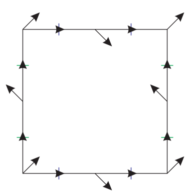
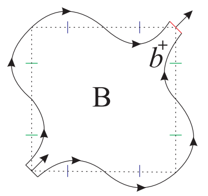


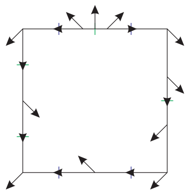
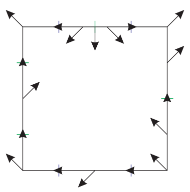
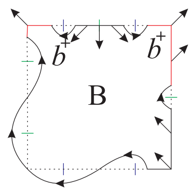
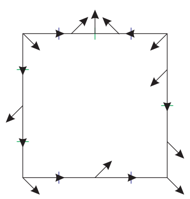
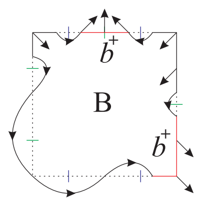
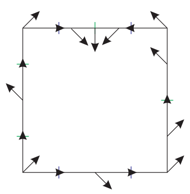
Figure 2-4: *Cases (a) - (g) for situation 1 with points on the boundary where* $f_1(\mathbf{y}) = 0$ *and* $f_2(\mathbf{y}) = 0$ *marked in green and blue respectively.*

Thus we have seven cases for which we must draw the generic vector fields on the boundary and work out the Conley index when we can. We will produce a table for each of these cases, showing the possible vector field around the edge of the rectangle which are unique up to reflection and rotation. Then we will construct isolating blocks for the case where it is possible and work out the Conley index.

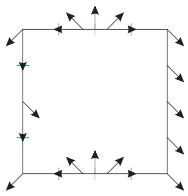
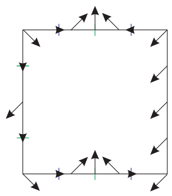
Case (a)

Direction of vector field round edge of rectangle.	Isolating block constructed from rectangle.	Conley index calculated from isolating block.
		$[([b^+],[b^+])] = \bar{0}$
		$[([b^+],[b^+])] = \bar{0}$

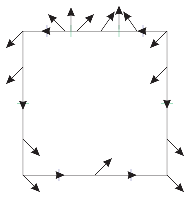
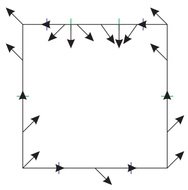
Case (b)

Direction of vector field round edge of rectangle.	Isolating block constructed from rectangle.	Conley index calculated from isolating block.
	No isolating block.	
		$[(S^1, [b^+])]$
		$[(S^1, [b^+])]$
	No isolating block.	

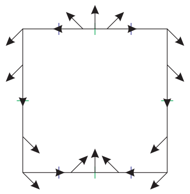
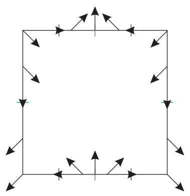
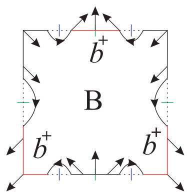
Case (c)

Direction of vector field round edge of rectangle.	Isolating block constructed from rectangle.	Conley index calculated from isolating block.
	No isolating block.	
	No isolating block.	

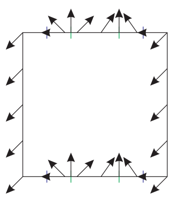
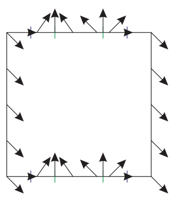
Case (d)

Direction of vector field round edge of rectangle.	Isolating block constructed from rectangle.	Conley index calculated from isolating block.
	No isolating block.	
	No isolating block.	

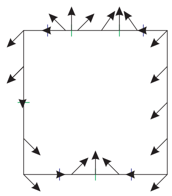
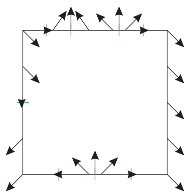
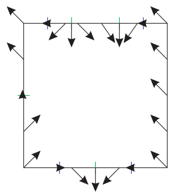
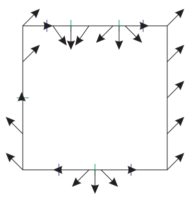
Case (e)

Direction of vector field round edge of rectangle.	Isolating block constructed from rectangle.	Conley index calculated from isolating block.
	No isolating block.	
		$[(S^1 \vee S^1, [b^+])]$

Case (f)

Direction of vector field round edge of rectangle.	Isolating block constructed from rectangle.	Conley index calculated from isolating block.
	No isolating block.	
	No isolating block.	

Case (g)

Direction of vector field round edge of rectangle.	Isolating block constructed from rectangle.	Conley index calculated from isolating block.
	No isolating block.	
	No isolating block.	
	No isolating block.	
	No isolating block.	

In the above cases we calculate the Conley index for the cases when we can form a isolating block for the generic vector field.

In cases where there are only two generic vector fields shown the other two vector fields are just rotations or reflections of one of the cases given.

Situation 2

In this situation the curve $f_2(\mathbf{y}) = 0$ intersects each side of the rectangle once and if we again consider the possible cases when the asymptotes of $f_1(\mathbf{y}) = 0$ are parallel to or not parallel to the asymptotes of $f_2(\mathbf{y}) = 0$, we get the cases given in figure 2-5.

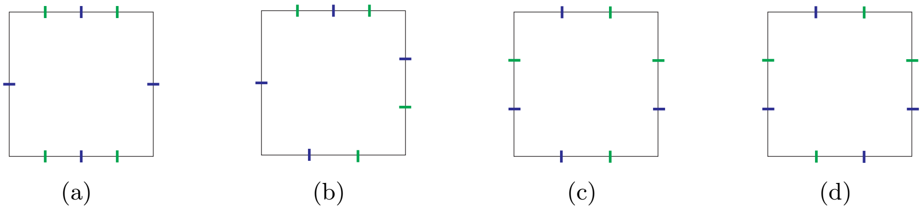
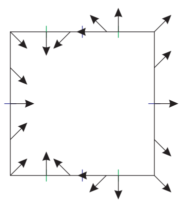
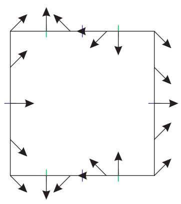
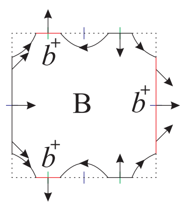


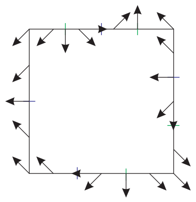
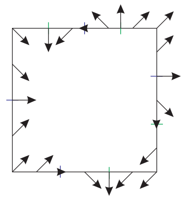
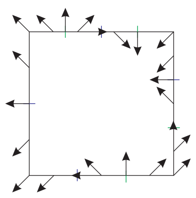
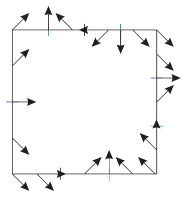
Figure 2-5: *Cases (a) - (d) for situation 2 with points on the boundary where $f_1(\mathbf{y}) = 0$ and $f_2(\mathbf{y}) = 0$ marked in green and blue respectively.*

Thus we have four cases, shown in figure 2-5, for which we must draw the generic vector fields and calculate the Conley index when we can. These calculations are found in the tables below, we only show the calculations for vector fields which are unique up to rotation and reflection.

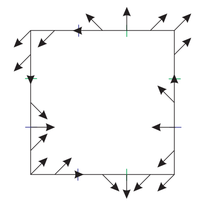
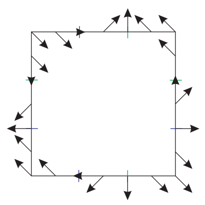
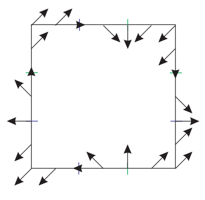
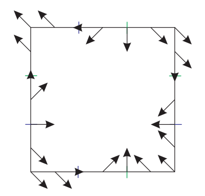
Case (a)

Direction of vector field round edge of rectangle.	Isolating block constructed from rectangle.	Conley index calculated from isolating block.
	No isolating block.	
		$[(S^1 \vee S^1, [b^+])]$

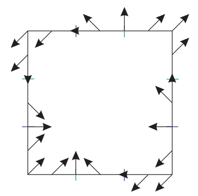
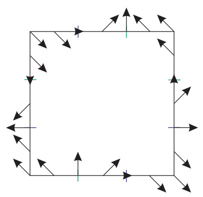
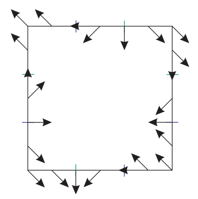
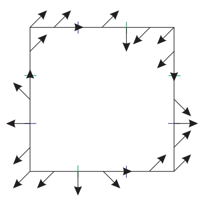
Case (b)

Direction of vector field round edge of rectangle.	Isolating block constructed from rectangle.	Conley index calculated from isolating block.
	No isolating block.	
	No isolating block.	
	No isolating block.	
	No isolating block.	

Case (c)

Direction of vector field round edge of rectangle.	Isolating block constructed from rectangle.	Conley index calculated from isolating block.
	No isolating block.	
	No isolating block.	
	No isolating block.	
	No isolating block.	

Case (d)

Direction of vector field round edge of rectangle.	Isolating block constructed from rectangle.	Conley index calculated from isolating block.
	No isolating block.	
	No isolating block.	
	No isolating block.	
	No isolating block.	

Again we are only able to calculate the Conley index for those cases where we can construct an isolating block for the generic vector fields.

In conclusion it can be seen that the Conley index takes values, $\overline{0}$, $[(S^1, [b^+])]$ or $[(S^1 \vee S^1, [b^+])]$, for the cases we can calculate.

$\square$

Remark 2.4.8. We saw, in the preceding lemma, that the idea of calculating the Conley index by looking at generic vector fields does not work very well for the case when $\det M < 0$ and $\det N < 0$. There are several possible reasons for this, firstly we may need more detailed information about the vector field in order to be able to construct the isolating block, secondly we may have a invariant set which it is not possible to isolate (such as if the phase plane contained only a pair of equilibria with one unstable and the other stable) or possibly we are in a degenerate case where asymptotes of the curves $f_1(\mathbf{y}) = 0$ and $f_2(\mathbf{y}) = 0$ line up and we have only one equilibrium.

Now, that we have calculated the possible Conley indices, we move onto the main point of this section proving: the existence of non-trivial solutions.

Remark 2.4.9. In what follow when we say that a equilibria is stable, unstable or a saddle equilibria we mean that the two eigenvalues of the linearisation about the equilibria both have negative real part, both have positive real part or one has negative real part and one has positive real part.

Proposition 2.4.10. *If the Conley index of an invariant set containing the equilibria of the ordinary differential equation* (2.33) *is* $\overline{0}$, $[(S^1, [b^+])]$ *or* $[(S^1 \vee S^1, [b^+])]$, *then the ordinary differential equation has a non-trivial solution connecting to* 0.

Proof: We prove this statement via a contradiction argument. Suppose that the equilibria 0 is an isolated invariant set. Then we can split the invariant set into two disjoint isolated invariant sets 0 and I, so by Smoller [45, Theorem 22.31] we have that

$$h\left(\{0\} \cup I\right) = h(0) \vee h(I).$$

However we know that the equilibria at 0 is either a stable or unstable equilibria by direct inspection of equation (2.32) and thus

$$h(0) = [(0 \sqcup b^+, [b^+])] \text{ or } [(S^2, b^+)].$$

Now $(0 \sqcup b^+) \vee h(I)$ and $S^2 \vee h(I)$ are not homotopic to $\overline{0}$, $[(S^1, [b^+])]$ or $[(S^1 \vee S^1, [b^+])]$, so the Conley index of an isolating block containing 0 and I is not equal to what the Conley index would be if 0 was isolated, which is a contradiction. Thus there must be a non-trivial solution which connects to zero, either as $\tau \to \infty$ or $\tau \to -\infty$.

□

Thus we have proved the existence of a non-trivial solution for the ordinary differential equation (2.33), however we do not know what type of solution we have found.

The Poincaré-Bendixson theorem tells us that this solution must be a periodic, heteroclinic or homoclinic orbit. In the next result we give conditions on the quadratic terms which ensure that the solution we have found will be a heteroclinic connection between two equilibria. For the convenience of the reader we recall an equivalent form of equation (2.33),

$$\frac{dy_1}{d\tau} = \lambda_1 y_1 - \lambda_2 y_2 + m_1 y_1^2 + 2m_2 y_1 y_2 + m_3 y_2^2$$
$$\frac{dy_2}{d\tau} = \lambda_2 y_1 + \lambda_1 y_2 + n_1 y_1^2 + 2n_2 y_1 y_2 + n_3 y_2^2.$$

Proposition 2.4.11. *Suppose that equation* (2.33) *has a non-trivial solution then, this solution will be a heteroclinic connection between two equilibria if the following conditions hold*

$$\lambda_1 m_2 + \lambda_2 n_2 = \lambda_1 n_1 - \lambda_2 m_1$$
$$\lambda_1 n_2 - \lambda_2 m_2 = \lambda_1 m_3 + \lambda_2 n_3$$

Proof: We prove this result by showing that under the above assumptions the ordinary differential equation (2.33) has a gradient structure i.e.

$$\mathbf{y}_\tau = \begin{pmatrix} \lambda_1 & -\lambda_2 \\ \lambda_2 & \lambda_1 \end{pmatrix} \nabla K(\mathbf{y})$$

for some function $K : \mathbb{R}^2 \to \mathbb{R}$. If this is the case then K will be an increasing or decreasing function along solutions. This then eliminates the possibility of periodic and homoclinic orbits, so by the Poincaré-Bendixson theorem the only thing the non-trivial solution can be is a heteroclinic connection between two equilibria. Thus all we need to do is construct K.

Now, for equation (2.33), we have

$$\begin{aligned}\mathbf{y}_\tau = &\begin{pmatrix} \lambda_1 & -\lambda_2 \\ \lambda_2 & \lambda_1 \end{pmatrix} \mathbf{y} + \begin{pmatrix} m_1 y_1^2 + 2m_2 y_1 y_2 + m_3 y_2^2 \\ n_1 y_1^2 + 2n_2 y_1 y_2 + n_3 y_2^2 \end{pmatrix} \\ = &\begin{pmatrix} \lambda_1 & -\lambda_2 \\ \lambda_2 & \lambda_1 \end{pmatrix} \Bigg(\mathbf{y} + \\ &\frac{1}{\lambda_1^2 + \lambda_2^2} \begin{pmatrix} (\lambda_1 m_1 + \lambda_2 n_1) y_1^2 + 2(\lambda_1 m_2 + \lambda_2 n_2) y_1 y_2 + (\lambda_1 m_3 + \lambda_2 n_3) y_2^2 \\ (\lambda_1 n_1 - \lambda_2 m_1) y_1^2 + 2(\lambda_1 n_2 - \lambda_2 m_2) y_1 y_2 + (\lambda_1 n_3 - \lambda_2 m_3) y_2^2 \end{pmatrix} \Bigg).\end{aligned}$$

Thus, using the assumption from the statement of the proposition, we can just take

$$\begin{aligned} K(\mathbf{y}) = &\frac{y_1^2 + y_2^2}{2} + \frac{1}{\lambda_1^2 + \lambda_2^2} \Bigg(\frac{(\lambda_1 m_1 + \lambda_2 n_1)}{3} y_1^3 + (\lambda_1 m_2 + \lambda_2 n_2) y_1^2 y_2 \\ &+ (\lambda_1 m_3 + \lambda_2 n_3) y_1 y_2^2 + \frac{(\lambda_1 n_3 - \lambda_2 m_3)}{3} y_2^3 \Bigg). \end{aligned}$$

Hence, under the assumptions of the proposition the equation has a gradient structure and the non-trivial solution must be a heteroclinic connection between two equilibria.

□

Remark 2.4.12. Furthermore, as heteroclinic connections between hyperbolic equilibria of which at least one is a stable or unstable equilibria, are structurally stable they persist under small perturbations. Thus there will exist a heteroclinic connections between equilibria connecting to 0 for equations which are close to equations satisfying the conditions of the preceding lemma. Hence we have an open set around the choices of nonlinearity which satisfy the conditions of the above lemma for which the solution we find on the centre manifold is a heteroclinic connection between equilibria.

Solutions on the Centre Manifold

In the previous section we showed that for the 2 dimensional real version of the unperturbed equation (2.32), we have the existence of a nontrivial bounded solution. This solution will correspond to a solution of the unperturbed complex ordinary differential equation (2.31).

We now want to show that a nontrivial solution exists for the perturbed complex ordinary differential equation (2.30). To see that this is the case note that by letting $\tilde{z}(\tilde{\tau}) = y_1(\tilde{\tau}) + \mathrm{i}y_2(\tilde{\tau})$ in (2.30) we can write this equation a two dimensional real ordinary differential equation, which will be a perturbation of (2.32). It then follows from lemma B.2.9 that the Conley indices we have calculated for (2.32) will persist for $|\delta|$ sufficiently small. Therefore the same arguments as where used in the previous section will give the existence of a nontrivial bounded solution. Hence it follows that the perturbed complex ordinary differential equation (2.30) will have a nontrivial solution.

On the otherhand if the condition of lemma 2.4.11 are satisfied then structural stability will give the persistence of a heteroclinic connection between 0 and a second equilibrium for $|\delta|$ sufficiently small.

Now if we denote the nontrivial solution to (2.30) by $\tilde{z}(\tilde{\tau})$, then if we reverse the blow-up rescaling we get a solution to the original complex ordinary differential equation (2.29)

$$z(\tau) = \delta\tilde{z}(\delta\tau),$$

for $\delta \in \mathbb{R}$ with $|\delta|$ sufficiently small. This solution then gives a solution to the equation on $X_c \times \mathbb{R}$

$$(U^c(\tau), \delta(\tau)) = \left(\delta\tilde{z}(\delta\tau)U_\eta^+ + \delta\overline{\tilde{z}}(\delta\tau)U_{-\eta}^+\right),$$

for $|\delta|$ sufficiently small.

Thus, to find a solution on the centre manifold, we just need to ensure that $(U^c(\tau) + \psi(U^c(\tau), \delta), \delta) \in \Omega$ for all $\tau \in \mathbb{R}$.

Now, since $\tilde{z}$ is a bounded solution, we have that

$$\sup_{\tau\in\mathbb{R}} \|U^c(\tau)\|_X \to 0 \text{ as } \delta \to 0.$$

Thus, as $\psi(0,0) = 0$ and ψ is continuous, it follows that for δ sufficiently small $(U^c(\tau) + \psi\left(U^c(\tau), \delta\right), \delta) \in \Omega$ for all $\tau \in \mathbb{R}$. Hence for sufficiently small δ there exist a non-trivial solution which is mapped on to the local centre manifold. Furthermore if the conditions of proposition 2.4.11 are satisfied then the solution will be a heteroclinic connections between possibly spatial periodic equilibria.

Hence, by proposition 2.3.1, for δ sufficiently small we have a non-trivial solution for the spatial dynamical system and we have completed the proof of the second part of theorem 2.1.1.

2.5 Conclusion

In this chapter we have proved the existence of generalised travelling wave solutions for the reaction diffusion equation in two cases.

For the first case we have shown that if we consider the reaction diffusion equation

$$u_t = \operatorname{div}\left(A\nabla u\right) + \delta u + p\left(\frac{x}{\varepsilon}\right) q(u),$$

then for all $\varepsilon > 0$ and $c \neq 0$ there exists a generalised travelling wave solution

$$u(x,t) = v(x \cdot k - ct, \frac{x}{\varepsilon}),$$

provided $\int_{T^2} p(s)\mathrm{d}s \neq 0$ and $|\delta| > 0$ is sufficiently small. Furthermore this generalised travelling wave solution is such that

$$v\left(\tau, \xi\right) \to v^{\pm}(\xi) \text{ as } \tau \to \pm\infty,$$

where the equilibria $v^{\pm}(\xi)$ are solutions $u(x) = v^{\pm}(x/\varepsilon)$ of the static problem

$$0 = \operatorname{div}\left(A\nabla u\right) + \delta u + p\left(\frac{x}{\varepsilon}\right) q(u),$$

one of which is the zero solution.

This result is slightly different to the usual travelling wave results which look at a fixed nonlinearity and find wave speeds c for which a travelling wave solution exists. What our result says is that for a given waves speed $c \neq 0$ there will exist a generalised travelling wave solution provided δ which is the coefficient of the linear terms is sufficiently close to zero. Alternatively as the non-zero equilibria tends to zero as $\delta \to 0$ this condition could be interpreted as saying that for $c \neq 0$ a generalised travelling wave solution exists provided the non-zero equilibria, it connects to zero, is sufficiently close to zero.

On the other hand in the second case we consider the reaction diffusion equation

$$u_t = \mathrm{div}\,(A\nabla u) + \left(\frac{1}{\varepsilon^2}\eta^T A\eta + \delta\right) u + p\left(\frac{x}{\varepsilon}\right) q(u),$$

and show that for all $\varepsilon > 0$ and $c \neq 0$ there exists a generalised travelling wave solution provided $\chi(\eta) = 2$, p satisfies certain conditions which are given in the proof or the technical result that follows this section and $\delta \neq 0$ with $|\delta|$ is sufficiently small. Also if we have additional conditions on p then the generalised travelling wave solution will have the same structure as those found in the first case.

For this case the addition of the term $\frac{1}{\varepsilon^2}\eta^T A\eta$ should be interpreted as shifting the spectrum of the divergence operator $\mathrm{div}\,(A\nabla u)$, as it is an eigenvalue of this operator. Thus in this second case we prove the existence of generalised travelling wave solutions for a reaction diffusion equation where the spectrum has been shifted.

2.6 Technical Result

In this section we give a more detailed version of the result presented in section 2.1.

Consider the reaction diffusion equation in $\mathbb{R}^2$ with a nonlinearity which is periodic in the spatial variable $x \in \mathbb{R}^2$;

$$u_t = \mathrm{div}\,(A\nabla u) + f\left(\frac{x}{\varepsilon}, u\right),$$

where A is a real symmetric positive definite matrix, $\varepsilon > 0$ and f is the nonlinearity. We assume that the nonlinearity f is of the form

$$f\left(\frac{x}{\varepsilon}, u\right) = \mu u p\left(\frac{x}{\varepsilon}\right) q\,(u)\,;$$

where $\mu \in \mathbb{R}$, $p \in H^2\left(T^2\right)$ and $q \in C^2\,(\mathbb{R})$ with $q(0) = 0$, $q'(0) = 0$, and $q''(0) \neq 0$. Furthermore let us write p and q in the following way $p(\xi) = \sum_{m\in\mathbb{Z}^2} p_m \exp\,(im \cdot \xi)$ and $q(s) = q_2 s^2 + O(s^3)$.

We look for generalised travelling solutions in a direction $k \in S_A^1$ of the form

$$u(x,t) = v\left(\frac{x}{\varepsilon}, x \cdot k - ct\right),$$

where $v = v(\tau, \xi)$ is periodic in ξ. Then treating μ as a parameter and allowing it to vary we get the following result.

Theorem 2.6.1. *Let $\varepsilon > 0$ and $c \neq 0$ be fixed then, if*

$$\mu = \frac{1}{\varepsilon^2}\eta^T A\eta + \delta$$

for $\eta \in \mathbb{Z}^2$, we have the following results.

- *If $\chi(\eta) := \#\left\{m \in \mathbb{Z}^2 : m^T Am = \eta^T A\eta\right\} = 1$ and $p_0 \neq 0$ then for $|\delta|$ sufficiently small there exists a generalised travelling wave corresponding to a heteroclinic connection between possibly spatially periodic equilibria i.e.*

$$v(\tau,\xi) \to v^{\pm}(\xi) \text{ as } \tau \to \pm\infty.$$

- *If $\chi(\eta) = 2$ and we define the constants*

$$d_1 = Re\left\{\frac{p_{-\eta} + 2p_\eta + p_{3\eta}}{\lambda_\eta^+ - \lambda_\eta^-}\right\} Re\left\{\frac{2p_\eta - p_{-\eta} - p_{3\eta}}{\lambda_\eta^+ - \lambda_\eta^-}\right\} - Im\left\{\frac{p_{-\eta} - p_{3\eta}}{\lambda_\eta^+ - \lambda_\eta^-}\right\}^2$$

and

$$d_2 = Im\left\{\frac{p_{-\eta} + 2p_\eta + p_{3\eta}}{\lambda_\eta^+ - \lambda_\eta^-}\right\} Im\left\{\frac{2p_\eta - p_{-\eta} - p_{3\eta}}{\lambda_\eta^+ - \lambda_\eta^-}\right\} - Re\left\{\frac{p_{3\eta} - p_{-\eta}}{\lambda_\eta^+ - \lambda_\eta^-}\right\}^2,$$

where

$$\lambda_\eta^{\pm} = -\frac{\left(c + \frac{2i}{\varepsilon}\eta^T Ak\right) \pm \sqrt{\left(c + \frac{2i}{\varepsilon}\eta^T Ak\right)^2 + \frac{4}{\varepsilon^2}\left(m^T Am - \eta^T A\eta\right)}}{2}$$

Then if d_1, d_2 are both positive or one positive and one negative and $|\delta|$ is sufficiently small, there exist a non-trivial generalised travelling wave solution. Furthermore if the conditions

$$\lambda_1 Im\left\{\frac{p_{-\eta} - p_{3\eta}}{\lambda_\eta^+ - \lambda_\eta^-}\right\} + \lambda_2 Re\left\{\frac{p_{3\eta} - p_{-\eta}}{\lambda_\eta^+ - \lambda_\eta^-}\right\} = \lambda_1 Im\left\{\frac{p_{-\eta} + 2p_\eta + p_{3\eta}}{\lambda_\eta^+ - \lambda_\eta^-}\right\} - \lambda_2 Re\left\{\frac{p_{-\eta} + 2p_\eta + p_{3\eta}}{\lambda_\eta^+ - \lambda_\eta^-}\right\}$$

and

$$\lambda_1 Re\left\{\frac{p_{3\eta} - p_{-\eta}}{\lambda_\eta^+ - \lambda_\eta^-}\right\} - \lambda_2 Im\left\{\frac{p_{-\eta} - p_{3\eta}}{\lambda_\eta^+ - \lambda_\eta^-}\right\} = \lambda_1 Re\left\{\frac{2p_\eta - p_{-\eta} - p_{3\eta}}{\lambda_\eta^+ - \lambda_\eta^-}\right\} + \lambda_2 Im\left\{\frac{2p_\eta - p_{-\eta} - p_{3\eta}}{\lambda_\eta^+ - \lambda_\eta^-}\right\},$$

where

$$\lambda_1 + i\lambda_2 = \frac{1}{c + \frac{2i}{\varepsilon}k^T A\eta}$$

hold, then the non-trivial solution will be a heteroclinic connection between two possibly spatially periodic equilibria.

The above result is a restriction of the result which was stated in theorem 2.1.1, with the conditions on the nonlinearity given explicitly. Hence the proof of this theorem is exactly the same as the proof of theorem 2.1.1.

Chapter 3

Slow Speed Travelling Waves for $\mu = \delta$

In the preceding chapter we considered the existence of generalised travelling waves for the reaction diffusion equation with a spatially periodic nonlinearity. We were able to show that if we fix the speed $c \neq 0$, then for sufficiently small $\mu = \delta$ there exists a generalised travelling wave solution corresponding to a heteroclinic connection between equilibria.

One of the key points about this result was that we first fixed $c \neq 0$ and then had to choose δ sufficiently small. So the size of δ depends on the size of c. This leads to the natural question: What happens if δ is not sufficiently small relative to c?

Our aim in this chapter is to investigate what happens as c and δ vary close to zero, this will be done by treating both c and δ as parameters and studying the bifurcations that occur.

3.1 Problem and Results

Consider a reaction diffusion equation in $\mathbb{R}^2$ with a nonlinearity which is periodic in the spatial variable $x \in \mathbb{R}^2$,

$$u_t = \operatorname{div}(A\nabla u) + f\left(\frac{x}{\varepsilon}, u\right), \tag{3.1}$$

where A is a real symmetric positive definite matrix, $\varepsilon > 0$ and f is the nonlinearity. We assume that the nonlinearity f is of the form,

$$f\left(\frac{x}{\varepsilon}, u\right) = \delta u + p\left(\frac{x}{\varepsilon}\right) q(u),$$

where $\delta \in \mathbb{R}$, $p \in H^2\left(T^2\right)$, the space of periodic Sobolev functions on $[0, 2\pi]^2$, and $q \in C^2(\mathbb{R})$ with $q(0) = 0$, $q'(0) = 0$ and $q''(0) \neq 0$.

We look for generalised travelling wave solutions in a direction $k \in S_A^2$ with speed c of the form,

$$u(x,t) = v\left(x \cdot k - ct, \frac{x}{\epsilon}\right), \tag{3.2}$$

where the profile function $v = v(\tau, \xi)$ is a function $v : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$, which is periodic in ξ with periodic cell $[0, 2\pi]^2$, i.e

$$v(\tau, \xi_1 + \xi_2) = v(\tau, \xi_1) \text{ for all } \xi_2 \in (2\pi\mathbb{Z})^2 .$$

We will study the existence of travelling wave solutions as c and δ vary around zero, this will be done by looking at the bifurcations which occur. From our analysis we will prove the following theorem.

Theorem 3.1.1. *Let, $|\delta|$, $|c|$ and $\varepsilon > 0$ be sufficiently small, and p be in a non-empty open set of $H^2(T^2)$, with $\int_{T^2} p(s)ds \neq 0$. Then there exists a generic bifurcation diagram in c and δ for the generalised travelling wave solution v (up to reflection in δ) which is shown in figure 3-1.*

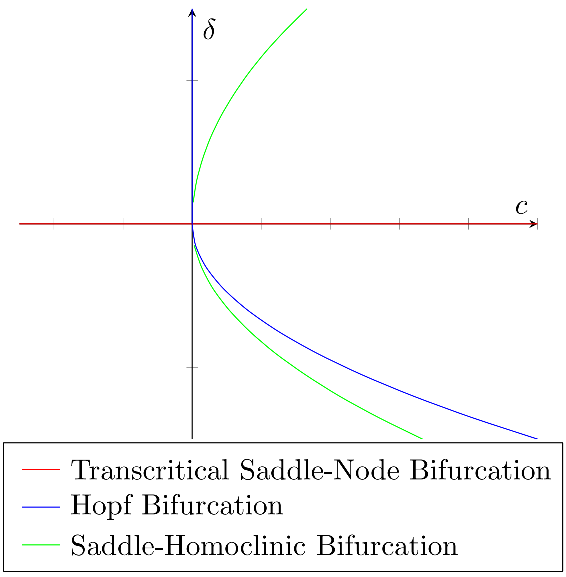


Figure 3-1: *Generic bifurcation diagram for generalised travelling wave solutions.*

Furthermore the bifurcations that occur produce generalised travelling wave solutions of different types.

1. *At the transcritical saddle-node bifurcation two equilibria exchange stability and we get a generalised travelling wave solution corresponding to a heteroclinic connection between equilibria. (These are the solutions found in Theorem 2.1.1.)*

2. *At the Hopf bifurcation we get the creation of a generalised travelling wave solution*

with a periodic structure in τ *i.e.*

$$v(\tau + T, \xi) = v(\tau, \xi),$$

for some $T > 0$*.*

3. *At the saddle-homoclinic bifurcation we get the creation of a generalised travelling wave solution with a homoclinic structure in* τ *i.e.*

$$v(\tau, \xi) \to v_0(\xi) \text{ as } \tau \to \pm\infty.$$

Remark 3.1.2.

- For a more detailed version of the above theorem where the conditions on p are given explicitly see section 3.6.
- When we say equilibria in the preceding theorem we mean equilibria in τ, they can be ξ dependent.
- The behaviour of this bifurcation is similar to that of the Bogdanov-Takens bifurcation, information about which can be found in the books by Kuznetsov [35] and by Guckenheimer and Holmes [21].

This result will be proved using a similar method to the one we used in chapter 2. Thus we start by formulating the problem as a spatial dynamical system. Then once we have done this we will perform a centre manifold reduction and study the dynamics on the centre manifold.

3.2 Spatial Dynamical System Formulation

In this section we will derive a spatial dynamical system formulation for the problem introduced in the preceding section.

Hence we start by substituting the ansatz

$$u(x, t) = v(x \cdot k - ct, \frac{x}{\varepsilon}), \tag{3.2}$$

where $v = v(\tau, \xi)$ is periodic in ξ ith periodic cell $[0, 2\pi^2]$, into the reaction diffusion equation (3.1) with $\mu = \delta$, to obtain an equation for the profile function v

$$-cv_\tau = \frac{1}{\varepsilon^2}\text{div}_\xi\,(A\nabla_\xi v) + \frac{2}{\varepsilon}k^T A\nabla_\xi v_\tau + v_{\tau\tau} + \delta v + p(\xi)q(v).$$

This equation is a second order elliptic partial differential equation for $(\tau, \xi) \in \mathbb{R} \times [0, 2\pi]^2$, with periodic boundary conditions on the cross-section.

We use the idea of Kirchgässner [33] to formulate this equation as a spatial dynamical system by treating τ as time. Since we want to investigate what happens as c and δ vary around zero, we treat c and δ as variable which are constant on solutions. Thus letting $U = (v, v_\tau)$ we obtain the spatial dynamical system

$$\begin{aligned} U_\tau &= \left(\hat{\mathcal{A}} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (\mathrm{Id} - P)\right) U + \nu \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (\mathrm{Id} - P)\, U + G(U, \delta, c) \\ \alpha_\tau &= 0, \end{aligned}$$

where

$$\begin{aligned} \hat{\mathcal{A}} &= \begin{bmatrix} 0 & 1 \\ -\frac{1}{\varepsilon^2}\mathrm{div}_\xi\,(A\nabla_\xi \cdot) & -\frac{2}{\varepsilon}k^T A\nabla_\xi \end{bmatrix}, \\ G(U, \delta, c) &= \begin{bmatrix} 0 \\ -\delta v - c v_\tau - pq(v) \end{bmatrix}, \\ \alpha &= (\delta, c, \nu) \end{aligned}$$

and

$$PU = \frac{1}{|T^2|}\int_{T^2} U(s)\mathrm{d}s.$$

The reason for the perturbation by the term,

$$(\nu - 1)\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (\mathrm{Id} - P)\, U,$$

where ν is an additional variable, is to ensure that we have a finite dimensional centre manifold. The idea is to look for solutions on the centre manifold such that $\nu \equiv 1$ and the perturbation is cancelled out. This idea was inspired by [24, Remark 3.6].

An alternative way to formulate the spatial dynamical system is the following

$$\begin{aligned} U_\tau &= \left(\mathcal{A} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} P\right) U + \nu \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (\mathrm{Id} - P)\, U + G(U, \delta, c) \qquad (3.3) \\ \alpha_\tau &= 0, \end{aligned}$$

where

$$\mathcal{A} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{\varepsilon^2}\mathrm{div}_\xi\,(A\nabla_\xi \cdot) & -1 - \frac{2}{\varepsilon}k^T A\nabla_\xi \end{bmatrix}.$$

The advantage of this formulation is that $\mathcal{A}$ is the linear operator from chapter 2, with $c = 1$, which we have studied in detail. For notational convenience we will define,

$$\mathcal{B} := \left(\mathcal{A} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} P \right).$$

Throughout the rest of this chapter we will work with this second formulation (3.3). Later in this chapter we will construct spaces of periodic functions X and Z such that $\mathcal{B} \in \mathcal{L}(X, Z)$ and $G \in C^2\left(X \times \mathbb{R}^2, X\right)$.

3.3 Centre Manifold Reduction

In this section we will prove the existence of a local centre manifold reduction for the spatial dynamical system (3.3) we introduced in the previous section.

For details about the local centre manifold theorem see Appendix A.

3.3.1 Result

We want to show the existence of a local centre manifold, thus we will prove the following result.

Proposition 3.3.1. *There exists a finite dimensional subspace $X_c \times \mathbb{R}^3 \subset X \times \mathbb{R}^3$ and a projection π_c onto X_c. Letting $X_h = (Id - \pi_c)(X)$ there exists a neighbourhood of the origin $\Omega \subset X \times \mathbb{R}^3$ and a map $\psi \in C^2\left(X_c \times \mathbb{R}^3, X_h\right)$, with $\psi(0,0) = 0$ and $D\psi(0,0) = 0$, such that if $(U^c, \alpha) : I \to X_c \times \mathbb{R}^3$ solves*

$$U^c_\tau = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} U^c + \pi_c G\left(U^c + \psi\left(U^c, \alpha\right), c, \delta\right),$$
$$\alpha_\tau = 0$$

for some interval $I \subset \mathbb{R}$, and $(U(\tau), \alpha(\tau)) = \left(U^c(\tau) + \psi\left(U^c(\tau), \alpha(\tau)\right), \alpha(\tau)\right) \in \Omega$ for all $\tau \in I$, then (U, α) solves

$$U_\tau = \mathcal{B}U + \nu \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (Id - P)\, U + G(U, \delta, c)$$
$$\alpha_\tau = 0.$$

Proof of Proposition 3.3.1

This proposition is proved in an almost identical way to which proposition 2.3.1 was proved in chapter 2. The main difference being that the linear part of the spatial dynamical system $\mathcal{B}$ has a 2-dimensional zero eigenspace, so we get a 2-dimensional centre space X_c.

Therefore rather than giving a complete proof, we will where possible state the changes to results from chapter 2 and indicate the adjustments that need to be made to the proofs.

This result will be proved in two steps, first we prove the existence of a local centre manifold for the restricted system, where the parameters are set equal to zero

$$U_\tau = \mathcal{B}U + G(U, 0, 0),$$

then we will extend this result to the full system.

Proof for the Restricted System

For clarity we state the exact result that can be proved for the restricted system.

Proposition 3.3.2. *There exists a finite dimensional subspace $X_c \subset X$ with a projection π_c onto X_c. Lettimg $X_h = (Id - \pi_c)(X)$, there exists a neighbourhood of the origin $\Omega \subset X \times \mathbb{R}^3$ and a map $\psi \in C^2(X_c, X_h)$, with $\psi(0) = 0$ and $D\psi(0) = 0$, such that if $U^c : I \to X_c$ solves,*

$$U^c_\tau = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} U^c + \pi_c G(U^c + \psi(U^c), 0, 0),$$

for some interval $I \subset \mathbb{R}$, and $U(\tau) = U^c(\tau) + \psi(U^c(\tau)) \in \Omega$ for all $\tau \in I$, then U solves

$$U_\tau = \mathcal{B}U + G(U, 0, 0).$$

This result follows directly from the local centre manifold theorem A.0.3 found in Appendix A. So to prove this result we need to check that hypotheses (H1)-(H3) hold.

Thus we start by defining the phase space we will work on. To do this we consider the eigenproblem,

$$\mathcal{B}U = \lambda U \text{ for } \lambda \in \mathbb{C}.$$

We look for eigenfunctions of the form

$$U = \begin{bmatrix} 1 \\ \lambda \end{bmatrix} \exp(\mathrm{i}m \cdot \xi) \text{ for } m \in \mathbb{Z}^2 \setminus \{0\}.$$

A function of this form will be an eigenfunction if λ satisfies the quadratic equation

$$\lambda^2 + \left(1 + \frac{2\mathrm{i}}{\varepsilon} k^T A m\right) \lambda + \frac{1}{\varepsilon^2} m^T A m = 0.$$

Thus we have eigenvalues and eigenfunctions

$$\lambda_m^{\pm} := \frac{-\left(1 + \frac{2\mathrm{i}}{\epsilon} k^T A m\right) \pm \sqrt{\left(1 + \frac{2\mathrm{i}}{\varepsilon} k^T A m\right)^2 + \frac{4}{\varepsilon^2} m^T A m}}{2}, \tag{3.4}$$

and

$$U_m^{\pm} := \begin{bmatrix} 1 \\ \lambda_m^{\pm} \end{bmatrix} \exp\left(\mathrm{i} m \cdot \xi\right) \text{ for } m \in \mathbb{Z}^2 \setminus \{0\}.$$

Finally we have the eigenvalues $\lambda_0^{\pm} = 0$ with associated eigenfunction and generalised eigenfunction

$$U_0^+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } U_0^- = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Now we define a Hilbert space which will be the phase space for our analysis, for this definition and throughout this chapter we will use the summing conventions

$$\sum_{m \in \mathbb{Z}^2} a_m^{\pm} := \sum_{m \in \mathbb{Z}^2} a_m^+ + \sum_{m \in \mathbb{Z}^2} a_m^-$$

and

$$\sum_{m \in \mathbb{Z}^2} \pm a_m^{\pm} := \sum_{m \in \mathbb{Z}^2} a_m^+ - \sum_{m \in \mathbb{Z}^2} a_m^-,$$

Thus we define

$$Z := \left\{ U = \sum_{m \in \mathbb{Z}^2} \alpha_m^{\pm} U_m^{\pm} : \overline{\alpha_m^{\pm}} = \alpha_{-m}^{\pm} \text{ and } \sum_{m \in \mathbb{Z}^2} \left|\alpha_m^{\pm}\right|^2 \left(1 + \left|\lambda_m^{\pm}\right|^4\right) < \infty \right\}$$

with the inner product

$$(U, V)_Z = \sum_{m \in \mathbb{Z}^2} \alpha_m^{\pm} \overline{\beta_m^{\pm}} \left(1 + \left|\lambda_m^{\pm}\right|^4\right),$$

for $U = \sum_{m \in \mathbb{Z}^2} \alpha_m^{\pm} U_m^{\pm}$ and $V = \sum_{m \in \mathbb{Z}^2} \beta_m^{\pm} U_m^{\pm}$. This space is actually the same space as in chapter 2, with a different, equivalent norm. Now we want to define the space X which

$$\mathcal{B} = \left(\mathcal{A} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} P \right)$$

maps into Z. The natural choice for X would be the domain of $\mathcal{B}$ with the graph norm, but for this to be a Banach space we need to check that $\mathcal{B}$ is closed.

Lemma 3.3.3. *$\mathcal{B} : D(\mathcal{B}) \to Z$ is a closed operator.*

Proof: To prove this result all we need to show is that

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} P : Z \to Z,$$

is a bounded operator. Then, since we know from lemma 2.3.7 that $\mathcal{A}$ is closed, the result follows with $D(\mathcal{B}) = D(\mathcal{A})$.

Thus let $U = \sum_{m\in\mathbb{Z}^2} \alpha_m^{\pm} U_m^{\pm} \in Z$ then

$$\left\| \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} PU \right\|_Z = \left\| \begin{bmatrix} 0 \\ \alpha_0^- \end{bmatrix} \right\|_Z \leqslant \|U\|_Z ,$$

so we have that

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} P \in \mathcal{L}(Z, Z),$$

and it follows that it is a closed operator and we have completed the proof.

□

Thus we take $X = D(\mathcal{B})$ with the graph norm. Now that we have defined the spaces we will work on, we can start checking the hypotheses of the local centre manifold theorem. Hence we start by checking (H1).

Lemma 3.3.4. *$\mathcal{B} \in \mathcal{L}(X, Z)$ and*

$$H(U, \delta, c, \nu) := \nu \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (Id - P)\, U + G(U, \delta, c) \in C^2\left(X \times \mathbb{R}^3, X\right).$$

Proof: Since $X = D(\mathcal{B})$ with the graph norm and $\mathcal{B}$ is closed it follows that $\mathcal{B} \in \mathcal{L}(X, Z)$. Now to check the second part of the lemma we write H in terms of the function F from chapter 2. Thus, we write

$$\nu \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (\mathrm{Id} - P)\, U + G(U, \delta, c) = \nu \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (\mathrm{Id} - P)\, U - c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} U + F(U, \delta),$$

where,

$$F(U, \delta) = F((u_1, u_2), \delta) = \begin{bmatrix} 0 \\ -\delta u_1 - pq(u_1) \end{bmatrix}.$$

Hence, since we know $F \in C^2(X \times \mathbb{R}, X)$, we can just differentiate the other parts of the function to get the regularity we require. Hence for $\underline{h} = ((h_1, h_2), h_3, h_4, h_5) \in$

$X \times \mathbb{R}^3$ and $\underline{g} = ((g_1, g_2), g_3, g_4, g_5) \in X \times \mathbb{R}^3$, we get

$$\begin{aligned} D_{(U,\alpha)} &\left(\nu \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (\mathrm{Id} - P)\, U - c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} U \right) \underline{h} \\ &= \nu \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (\mathrm{Id} - P) \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} - c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \\ &\quad + h_5 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (\mathrm{Id} - P)\, U - h_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} U, \end{aligned}$$

and

$$\begin{aligned} D^2_{(U,\alpha)} &\left(\nu \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (\mathrm{Id} - P)\, U - c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} U \right) (\underline{h}, \underline{g}) \\ &= g_5 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (\mathrm{Id} - P) \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} - g_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \\ &\quad + h_5 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (\mathrm{Id} - P) \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} - h_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}. \end{aligned}$$

Thus we have the desired regularity.

□

Thus with the observation that $F(0,0) = 0$ and $D_{(U,\delta)}F(0,0) = 0$ we have verified hypothesis (H1) with $k = 2$. Now to check hypothesis (H2) we need some properties of the spectrum of $\mathcal{B}$, which are summarised in the following lemma.

Lemma 3.3.5. *For $\lambda_m^{\pm}$ as defined in (3.4) and $\lambda_0^{\pm} = 0$, we have that $|Re\lambda_m^{\pm}| \to \infty$ as $|m| \to \infty$, $\sigma(\mathcal{B}) = \left\{\lambda_m^{\pm} : m \in \mathbb{Z}^2\right\}$, the spectrum has gaps either side of the imaginary axis and the only eigenvalue with zero real part are $\lambda_0^{\pm} = 0$.*

Proof: The proof that $|\mathrm{Re}\lambda_m^{\pm}| \to \infty$ is identical to the proof of lemma 2.3.11 with c set to be equal to 1. Once we have this spectral growth the argument to show $\sigma(\mathcal{B}) = \left\{\lambda_m^{\pm} : m \in \mathbb{Z}^2\right\}$ and that the spectrum has a gap either side of the imaginary axis is identical to the one given for lemma 2.3.13.

Thus it just remains to check that the only eigenvalues with zero real part are $\lambda_0^{\pm}$. To prove this all we need to check is that $\mathrm{Re}\lambda_m^{\pm} \neq 0$ for all $m \in \mathbb{Z}^2 \setminus \{0\}$. Suppose for a contradiction that there exists one of these eigenvalues $\lambda \in \left\{\lambda_m^{\pm} : m \in \mathbb{Z}^2 \setminus \{0\}\right\}$ with zero real part. Then λ will solve the quadratic equation

$$\lambda^2 + \left(1 + \frac{2\mathrm{i}}{\varepsilon} k^T A m\right) - \frac{1}{\varepsilon^2} m^T A m = 0 \text{ for some } m \in \mathbb{Z}^2 \setminus \{0\}.$$

Let $\tilde{\lambda}$ be the other root of this quadratic equation then from the properties of roots we have

$$\lambda\tilde{\lambda} = \frac{1}{\varepsilon^2} m^T A m,$$
$$\lambda + \tilde{\lambda} = -\left(1 + \frac{2\mathrm{i}}{\varepsilon} k^T A m\right)$$

Thus, since the real part of λ is zero, the first equations implies that either $\mathrm{Re}\tilde{\lambda} = 0$ or $\mathrm{Im}\lambda = 0$. However the second equation tells use that $\mathrm{Re}\tilde{\lambda} \neq 0$; hence $\mathrm{Im}\lambda = 0$ and it follows that $\lambda = 0$. Then the first equation implies that $m^T A m = 0$ which is a contradiction, since $m \in \mathbb{Z}^2 \setminus \{0\}$. Hence the only eigenvalues with zero real part are $\lambda_0^{\pm} = 0$.

□

With these spectral properties we are now in a position to define X_c, but before we do that we define some sets and spaces of eigenfunctions. We define

$$\begin{aligned} \mathcal{S} &= \left\{U_m^{\pm} | m \in \mathbb{Z}^2\right\}, & S &= \mathrm{span}_{\mathbb{C}}\left\{\mathcal{S}\right\} \cap Z, \\ \mathcal{S}^s &= \left\{U_m^{\pm} | \mathrm{Re}\lambda_m^{\pm} < 0\right\}, & S^s &= \mathrm{span}_{\mathbb{C}}\left\{\mathcal{S}^s\right\} \cap Z, \\ \mathcal{S}^u &= \left\{U_m^{\pm} | \mathrm{Re}\lambda_m^{\pm} > 0\right\}, & S^u &= \mathrm{span}_{\mathbb{C}}\left\{\mathcal{S}^u\right\} \cap Z, \\ \mathcal{S}^c &= \left\{U_m^{\pm} | \mathrm{Re}\lambda_m^{\pm} = 0\right\}, & S^c &= \mathrm{span}_{\mathbb{C}}\left\{\mathcal{S}^c\right\} \cap Z; \end{aligned}$$

where the last three lines correspond to the sets and spans of stable, unstable and centre eigenfunctions. Now we define

$$X_c = Z_c := S_c = \mathrm{span}\left\{U_0^+, U_0^-\right\} = \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right).$$

Furthermore let Z_s, Z_u and Z_h denote the closures of S^s, S^u and $S^h := S^s \cup S^u$ in Z. Then we have that

$$Z = Z_s \oplus X_c \oplus Z_u = X_c \oplus Z_h.$$

Now to complete the verification of hypothesis (H2) we just need to define a projection onto X_c which commutes with $\mathcal{B}$. We define this projection in the following way, let $\tilde{\pi}_c$ be the projection of S onto S_c, defined in the natural way

$$\tilde{\pi}_c\left(\sum_{m\in\mathbb{Z}} \alpha_m^{\pm} U_m^{\pm}\right) = \alpha_0^+ U_0^+ + \alpha_0^- U_0^- \text{ for all } \sum_{m\in\mathbb{Z}} \alpha_m^{\pm} U_m^{\pm} \in S.$$

Then for each $U \in Z$ we define the projection onto X_c to be,

$$\pi_c U = \lim_{n\to\infty} \tilde{\pi}_c U_n$$

where $U_n \in S$ is a sequence which converges to U in Z. Then we have the following result about π_c.

Lemma 3.3.6. *π_c is well-defined, $\pi_c \in \mathcal{L}(Z, X)$ and π_c commutes with $\mathcal{B}$.*

Proof: The proof that π_c is well-defined and $\pi_c \in \mathcal{L}(Z, X)$ is identical to the proof of lemma 2.3.16, with the operator $\mathcal{A}$ replaced with $\mathcal{B}$. To show that π_c computes with $\mathcal{B}$ we use the same argument as lemma 2.3.17, with $\mathcal{A}$ again replaced by $\mathcal{B}$.

□

Hence we have completed the verification of hypothesis (H2). Notice that we can construct projections π_s, π_u and $\pi_h \in \mathcal{L}(Z)$ and $\mathcal{L}(X)$ onto Z_s, Z_u and Z_h which commute with $\mathcal{B}$, in a similar way to which π_c was constructed.

Final we need to check the final hypothesis (H3), again this is done in an almost identical way to chapter 2. Thus we start by constructing exponentially decaying semigroups on Z_s and Z_u.

Lemma 3.3.7. *Let $X_s = D(\mathcal{B}) \cap Z_s$, $X_u = D(\mathcal{B}) \cap Z_u$, $\mathcal{B}_s := \mathcal{B}|_{X_s}$ and $\mathcal{B}_u := \mathcal{B}|_{X_u}$. Then $\mathcal{B}_s$ and $-\mathcal{B}_u$ are closed operators which generate exponentially decaying C_0-semigroups of contractions on Z_s and Z_u with common decay constant $\gamma := \min\{|Re\lambda_m^{\pm}| : Re\lambda_m^{\pm} \neq 0\} > 0$.*

Proof: The argument to prove this result is identical to the proof of lemma 2.3.19, with $\mathcal{A}$ replaced with $\mathcal{B}$.

□

Notation 3.3.8. The C_0-semigroups generated by $\mathcal{B}_s$ and $\mathcal{B}_u$ will be denoted by $T_s(\tau)$ and $T_u(\tau)$.

We now use these semigroups to prove hypothesis (H3).

Lemma 3.3.9. *Let $X_h := X_s \oplus X_u$ then for each $\eta \in [0, \gamma)$ and $f \in C_\eta(\mathbb{R}, X_h)$, the space of exponentially growing continuous functions, the affine problem,*

$$U_\tau^h = \mathcal{B}U^h + f \text{ and } U^h \in C_\eta(\mathbb{R}, X_h), \tag{3.5}$$

has a unique solution $U^h = K_h f$, where $K_h \in \mathcal{L}(C_\eta(\mathbb{R}, X_h))$ and

$$\|K_h\|_{\mathcal{L}(C_\eta(\mathbb{R}, X_h))} \leqslant \Gamma(\eta);$$

for some continuous function $\Gamma : [0, \gamma) \to \mathbb{R}^+$.

Proof: Following the same argument as the proof of lemma 2.3.21 from chapter 2 we find that the affine problem (3.5) has a unique solution

$$K_h f(\tau) = \int_{-\infty}^{\tau} T_s(\tau - \sigma)\pi_s f(\sigma)\mathrm{d}\sigma - \int_{\tau}^{\infty} T_u(\sigma - \tau)\pi_u f(\sigma)\mathrm{d}\sigma,$$

with the estimate

$$\|K_h\|_{\mathcal{L}(C_\eta(\mathbb{R},X_h))} \leqslant \frac{2}{\gamma - \eta}.$$

Thus we have proved the lemma.

□

Hence we have verified the hypotheses of the local centre manifold theorem and proposition 3.3.2 follows directly from theorem A.0.3 with the observation that for $U^c \in X_c$

$$\mathcal{B}U^c = \left(\mathcal{A} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) U^c = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} U^c.$$

Proof for Full System

This local centre manifold reduction is then extended to the full system using an almost identical argument to the one given in chapter 2. The only difference is that there are three extra real variables instead of one. Thus we have completed the proof of proposition 3.3.1.

□

Thus we have completed the proof of the local centre manifold reduction. In the next section we will work out the ordinary differential equation which governs the dynamics on the centre manifold and study its bifurcations. But before we do this, we need to check that, for appropriate conditions, the open neighbourhood Ω from proposition 3.3.1 will be large enough that we can set $\nu = 1$, to ensure that we are solving the system we are interested in.

Lemma 3.3.10. *For any $r > 0$ there exists an $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$ then $B_r\left(0, X \times \mathbb{R}^3\right) \subset \Omega$.*

Proof: In order to prove this result we need to look at how the local centre manifold theorem is proved. The idea when proving the local centre manifold theorem is to first prove that if we have a Lipschitz continuous nonlinearity with sufficiently small Lipschitz constant then we have a global centre manifold [47, Theorem 2].

From this result the local centre manifold theorem is proved by multiplying the C^k-nonlinearity by a smooth cut-off function in such a way that it becomes a Lipschitz function with sufficiently small Lipschitz constant. Then for this modified nonlinearity we have a global centre manifold which will be a local centre manifold for the original system where the cut off function is equal to 1. Hence the open neighbourhood of the origin Ω is the open set where these two nonlinearities agree, i.e. where the cut off function is equal to 1.

Thus the size of Ω is determined by how large the Lipschitz constant is allowed to be and the way that the nonlinearity grows. Hence we want to find a bound on the size of the Lipschitz constant.

From the proofs of [47, Theorem 1 and 2] we see that, since we want a C^2 centre manifold, the bound on the size of the Lipschitz constant δ_2 is determined in the following way. If we define $K \in \mathcal{L}(C_\eta(\mathbb{R}, X))$ by

$$Kf(t) = \int_0^t \exp\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}(t-\sigma)\right)\pi_c f(\sigma)\mathrm{d}\sigma + K_h(\pi_h f)(t),$$

for $f \in C_\eta(\mathbb{R}, X)$. Then there exist a function $\Gamma_c : (0,\gamma) \to \mathbb{R}_+$ such that,

$$\|K\|_{\mathcal{L}(C_\eta(\mathbb{R},X))} \leqslant \Gamma_c(\eta) \text{ for all } \eta \in (0,\gamma),$$

and,

$$\delta_2 := \sup_{\eta\in(0,\frac{\gamma}{2})} \inf_{\xi\in(\eta,2\eta)} \Gamma_c(\xi)^{-1}.$$

Thus we want to find a bound for $\|K\|_{\mathcal{L}(C_\eta(\mathbb{R},X))}$ in terms of η. We know from the proof of lemma 3.3.9 that

$$\|K_h\|_{\mathcal{L}(C_\eta(\mathbb{R},X_h))} \leqslant \frac{2}{\gamma-\eta}.$$

Now if $f = (f_1, f_2) \in C_\eta(\mathbb{R}, X)$ then

$$\left\|\int_0^t \exp\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}(t-\sigma)\right)\pi_c f(\sigma)\mathrm{d}\sigma\right\|_X \leqslant \left\|\int_0^t \pi_c f(\sigma)\mathrm{d}\sigma\right\|_X + \left\|\int_0^t \begin{bmatrix} (t-\sigma)f_2(\sigma) \\ 0 \end{bmatrix}\mathrm{d}\sigma\right\|_X,$$

which with a bit of calculation, using the fact $\|\pi_c f(t)\|_X \leqslant \|f\|_\eta \mathrm{e}^{\eta|t|}$ and integration by parts on the second term, gives that,

$$\left\|\int_0^t \exp\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}(t-\sigma)\right)\pi_c f(\sigma)\mathrm{d}\sigma\right\|_X \leqslant \left(\frac{1+\eta}{\eta^2}\right)\|f\|_\eta \mathrm{e}^{\eta|t|}. \tag{3.6}$$

So it follows that

$$\|K\|_{\mathcal{L}(C_\eta(\mathbb{R},X))} \leqslant \frac{2}{\gamma-\eta} + \frac{1+\eta}{\eta^2}.$$

Now if we let $\eta = \gamma/4$ then

$$\sup_{\eta \in (0, \frac{\gamma}{2})} \inf_{\xi \in (\eta, 2\eta)} \Gamma_c(\xi)^{-1} \geqslant \inf_{\xi \in (\frac{\gamma}{4}, \frac{\gamma}{2})} \frac{\xi^2(\gamma - \xi)}{\xi^2 + (\gamma - 1)\xi + \gamma} \geqslant \min\left\{\frac{3\gamma^2}{20\gamma + 48}, \frac{3\gamma^2}{48\gamma + 32}\right\},$$

and thus

$$\delta_2 \geqslant \min\left\{\frac{3\gamma^2}{20\gamma + 48}, \frac{3\gamma^2}{48\gamma + 32}\right\} \to \infty \text{ as } \gamma \to \infty.$$

Thus the upper bound δ_2 on the size of the Lipschitz constant tends to infinity as the size of the spectral gap γ tends to infinity.

Now from the proof of the local centre manifold theorem [47, Theorem 3] we see that the size of Ω is determined by how large the Lipschitz constant is allowed to be, since we cut off the nonlinearity to ensure that the Lipschitz constant is smaller than δ_2. Thus the larger the Lipschitz constant is allowed to be the more of the nonlinearity we can include. Thus if $r > 1$ then $B_r\left(0, X \times \mathbb{R}^3\right) \subset \Omega$ for sufficiently large δ_2.

Hence to complete the proof all we need to show is that the spectral gap $\gamma \to \infty$ as $\varepsilon \to 0$. Using the formula for the non-zero eigenvalues (3.4) we see that for each $m \neq 0$,

$$\begin{aligned}
\operatorname{Re}\left\{\lambda_m^{\pm}\right\} &= \frac{1}{2}\left(-c \pm \operatorname{Re}\sqrt{\left(c + \frac{2\mathrm{i}}{\varepsilon}k^T A m\right)^2 + \frac{4}{\varepsilon^2} m^T A m}\right) \\
&= \frac{1}{2}\left(-c \pm \operatorname{Re}\sqrt{c^2 + \frac{4}{\varepsilon^2}\left(m^T A m - (k^T A m)^2\right) + \frac{4c\mathrm{i}}{\varepsilon}k^T A m}\right) \to \infty,
\end{aligned}$$

as $\varepsilon \to 0$. Now let $k_\perp \in \mathbb{R}^2$ be such that $\{k, k_\perp\}$ is a orthonormal basis for $\mathbb{R}^2$ with respect to the inner product $(x, y)_A = x^T A y$. Then it follows that

$$m = (m, k)_A k + (m, k_\perp) k_\perp$$

and

$$m^T A m = (m, k)_A^2 + (m, k_\perp)_A^2.$$

So for $m \neq 0$ we have

$$\begin{aligned}
&\left|\mathrm{Re}\left\{\lambda_m^{\pm}\right\}\right| \\
&\quad \geqslant \left| -|c| + \mathrm{Re}\sqrt{c^2 + \frac{4}{\varepsilon^2}(m,k_\perp)_A^2 + \frac{4c\mathrm{i}}{\varepsilon}(m,k)_A} \right| \\
&\quad \geqslant \left| -|c| + \frac{1}{\sqrt{2}}\sqrt{\sqrt{\left(c^2 + \frac{4}{\varepsilon^2}(m,k_\perp)_A^2\right)^2 + \left(\frac{4c}{\varepsilon}(m,k)_A\right)^2} + \left(c^2 + \frac{4}{\varepsilon^2}(m,k_\perp)_A^2\right)} \right| \\
&\quad \geqslant \left| -|c| + \frac{1}{\sqrt{2}}\sqrt{\sqrt{c^4 + \frac{8c^2}{\varepsilon^2}m^T A m} + c^2} \right|.
\end{aligned}$$

Now, since $m^T A m \to \infty$ as $|m| \to \infty$, there exists a $m_\star \in \mathbb{Z}^2 \setminus \{0\}$ such that $m_\star^T A m_\star = \min_{m \in \mathbb{Z}^2\setminus\{0\}} m^T A m$. Hence if follows that

$$\left|\mathrm{Re}\left\{\lambda_m^{\pm}\right\}\right| \geqslant \left| -|c| + \frac{1}{\sqrt{2}}\sqrt{\sqrt{c^4 + \frac{8c^2}{\varepsilon^2}m_\star^T A m_\star} + c^2} \right| \to \infty \text{ as } \varepsilon \to \infty.$$

Thus the size of the spectral gap $\gamma \to \infty$ as $\varepsilon \to 0$ and hence there exist a $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$ then $B_r\left(0, X \times \mathbb{R}^3\right) \subset \Omega$.

□

Thus we see that for $\varepsilon > 0$ sufficiently small Ω will be large enough that we can set $\nu = 1$ to eliminate the perturbation in the spatial dynamical system (3.3).

3.4 Dynamics on the centre manifold

In the previous section we showed that close to zero, we can study the dynamics of the spatial dynamical system (3.3) by studying the dynamics on the centre manifold.

In this section we will start by deriving the ordinary differential equation which governs the dynamics on the centre manifold. We will then study the bifurcations of this ordinary differential equation close to $(0, 0)$ as c and δ vary. Furthermore these bifurcations will create the types of solutions listed in theorem 3.1.1.

The first step towards studying the dynamics on the centre manifold is to calculate the ordinary differential equation on $X_c \times \mathbb{R}^3$ up to cubic order in U^c. We calculate the terms up to cubic order because they allows us to fully understand the bifurcations. Since we want to work out the terms of the ordinary differential equation on $X_c \times \mathbb{R}^3$ up to cubic order, we need to calculate the Taylor expansion of the reduction map ψ up to quadratic order in U^c.

Calculation of the Reduction Map

We calculate the Taylor expansion of the reduction map using a similar method to the one we used in chapter 2. Thus we will expand the reduction map in terms of eigenfunctions and solve the equation

$$\begin{aligned} D_{(U^c,\alpha)}\psi\left(U^c,\alpha\right) & \left(\begin{array}{c} \mathcal{B}U^c + \pi_c G\left(U^c + \psi\left(U^c,\alpha\right),\delta,c\right) \\ 0 \end{array}\right) \\ &= \mathcal{B}\psi\left(U^c,\alpha\right) + \pi_h\left(\nu\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\psi\left(U^c,\alpha\right) + G\left(U^c + \psi\left(U^c,\alpha\right),\delta,c\right)\right), \quad (3.7) \end{aligned}$$

on each component up to quadratic order in U^c. However, unlike in chapter 2, we will use an expansion in terms of the eigenfunctions of an auxiliary operator, which includes the c and δ parts, to simplify the calculation.

Hence we define the operator

$$\mathcal{C} := \begin{bmatrix} 0 & 1 \\ -\frac{1}{\varepsilon^2}\operatorname{div}_\xi\left(A\nabla_\xi\cdot\right) - \delta & -c - \frac{2\mathrm{i}}{\varepsilon}k^T A\nabla_\xi \end{bmatrix},$$

this operator has eigenvalues and eigenfunctions,

$$\mu_m^\pm := \frac{-\left(c + \frac{2\mathrm{i}}{\varepsilon}k^T Am\right) \pm \sqrt{\left(c + \frac{2\mathrm{i}}{\varepsilon}k^T Am\right)^2 + \frac{4}{\varepsilon^2}m^T Am - 4\delta}}{2}$$

and

$$V_m^\pm := \begin{bmatrix} 1 \\ \mu_m^\pm \end{bmatrix}\exp\left(\mathrm{i}m\cdot\xi\right) \text{ for } m \in \mathbb{Z}^2.$$

The functions $\left\{V_m^\pm : m \in \mathbb{Z}^2\right\}$ form an alternative Hilbert basis for $X_\mathbb{C}$ and $Z_\mathbb{C}$, the complexifications of X and Z.

Throughout this section we will need to use expansions of the function p and q, that appear in the nonlinearity. Thus, since $p \in H^2\left(T^2\right)$ and $q \in C^\infty\left(T^2\right)$ with $q(0) = 0$, $q'(0) = 0$ and $q''(0) \neq 0$, we can write $p(\xi) = \sum_{m\in\mathbb{Z}^2} p_m \mathrm{e}^{im\cdot\xi}$ and $q(s) = q_2 s^2 + O(s^3)$.

To calculate the terms of the reduction map, we set $\mu = 1$ and expand ψ in terms of the Hilbert basis $\left\{V_m^\pm : m \in \mathbb{Z}^2\right\}$. Thus we write,

$$\psi\left(U^c,\alpha\right) = \sum_{m\in\mathbb{Z}^2\setminus\{0\}} \psi_m^\pm\left(U^c,\alpha\right)V_m^\pm.$$

Now we want to derive the equation on each eigenfunction component up to quadratic order in U^c. To achieve this we first simplify and rearrange (3.7), using the fact

$$\left(\mathcal{A} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} P\right) U^c + \pi_c \left(\begin{bmatrix} 0 & 0 \\ -\delta & -c \end{bmatrix} \left(U^c + \psi\left(U^c, \alpha\right)\right)\right) = \begin{bmatrix} 0 & 1 \\ -\delta & -c \end{bmatrix} U^c$$

and

$$\left(\mathcal{A} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} P\right) \psi\left(U^c, \alpha\right) + \pi_h \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \left(\mathrm{Id} - P\right)\left(U^c + \psi\left(U^c, \alpha\right)\right) + \begin{bmatrix} 0 & 0 \\ -\delta & -c \end{bmatrix} \left(U^c + \psi\left(U^c, \alpha\right)\right)\right) = \mathcal{C}\psi\left(U^c, \alpha\right),$$

to get

$$D_{(U^c,\alpha)}\psi\left(U^c, \alpha\right) \begin{pmatrix} \begin{bmatrix} 0 & 1 \\ -\delta & -c \end{bmatrix} U^c + \pi_c J\left(U^c + \psi\left(U^c, \alpha\right)\right) \\ 0 \end{pmatrix} = \mathcal{C}\psi\left(U^c, \alpha\right) + \pi_h J\left(U^c + \psi\left(U^c, \alpha\right)\right), \tag{3.8}$$

where

$$J\left(\left(u_1, u_2\right)\right) = \begin{bmatrix} 0 \\ -pq\left(u_1\right) \end{bmatrix}.$$

Then, since we want the equations on each component upto quadratic order in U^c, we need to work out the terms of J upto quadratic order in U^c on X_c and X_h. Therefore if we let $U^c = (u_1^c, u_2^c)$ and $\psi_0^+\left(U^c, \alpha\right) + \psi_0^-\left(U^c, \alpha\right) = u_1^c$, then suppressing the arguments of the functions $\psi_m^\pm$ and substituting the expansion for ψ into J we get

$$\pi_c J\left(U^c + \psi\left(U^c, \alpha\right)\right) = \begin{bmatrix} 0 \\ -q_2 \sum_{l,n\in\mathbb{Z}^2} p_{-n-l}\left(\psi_l^+ + \psi_l^-\right)\left(\psi_n^+ + \psi_n^-\right) \end{bmatrix} + O\left(\|U^c\|^3\right)$$

and

$$\pi_h J\left(U^c + \psi\left(U^c, \alpha\right)\right) = q_2 \sum_{m\in\mathbb{Z}^2\setminus\{0\}} \mp \left(\sum_{l,n\in\mathbb{Z}^2} \frac{p_{m-l-n}\left(\psi_l^+ + \psi_l^-\right)\left(\psi_n^+ + \psi_n^-\right)}{\mu_m^+ - \mu_m^-}\right) V_m^\pm + O\left(\|U^c\|^3\right)$$

We first want to find the linear terms of the reduction map, so we write

$$\psi\left(U^c, \alpha\right) = \sum_{m\in\mathbb{Z}^2\setminus\{0\}} L_m^\pm(\delta, c) U^c V_m^\pm + O\left(\|U^c\|^2\right),$$

where the functions $L_m^\pm(\delta, c) : X_c \to \mathbb{C}$ are linear maps.

Now substituting this expansion into equation (3.8) we get that

$$\sum_{m\in\mathbb{Z}^2\setminus\{0\}} L_m^{\pm}(\delta,c)\left(\begin{bmatrix} 0 & 1 \\ -\delta & -c \end{bmatrix} U^c\right) V_m^{\pm} = \sum_{m\in\mathbb{Z}^2\setminus\{0\}} \mu_m^{\pm} L_m^{\pm}(\delta,c) U^c V_m^{\pm} + O\left(\|U^c\|^2\right),$$

which is solved, up to linear order, on each component if we set $L_m^{\pm}(\delta,c) = 0$ for all $m \in \mathbb{Z}\setminus\{0\}$, thus the reduction map has no linear terms.

In order to calculate the quadratic terms we let

$$\psi\left(U^c,\alpha\right) = \sum_{m\in\mathbb{Z}^2\setminus\{0\}} Q_m^{\pm}\left(U^c,\delta,c\right) V_m^{\pm} + O\left(\|U^c\|^3\right),$$

where $Q_m^{\pm}(\cdot,\delta,c) : X_c \to \mathbb{C}$ is a quadratic function. Now, plugging this expansion into (3.8), and using the formulas for the projection of J onto X_c and X_h upto quadratic order, we get

$$\begin{aligned}\sum_{m\in\mathbb{Z}^2\setminus\{0\}} & D_{U^c} Q\left(U^c,\delta,c\right)\left(\begin{bmatrix} 0 & 1 \\ -\delta & -c \end{bmatrix} U^c\right) V_m^{\pm} \\ &= \sum_{m\in\mathbb{Z}^2\setminus\{0\}} \mu_m^{\pm} Q_m^{\pm}\left(U^c,\delta,c\right) V_m^{\pm} + \sum_{m\in\mathbb{Z}^2\setminus\{0\}} \mp \frac{p_m q_2 \left(u_1^c\right)^2}{\mu_m^+ - \mu_m^-} V_m^{\pm} + O\left(\|U^c\|^3\right).\end{aligned}$$

Then to calculate the quadratic terms explicitly we let $U^c = \mathbf{y} = (y_1,y_2)^T$ and set

$$Q_m^{\pm}\left(\mathbf{y},\delta,c\right) = \mathbf{y}^T A_m^{\pm} \mathbf{y},$$

where $A_m^{\pm} \in \mathbb{R}^{2\times 2}$ is symmetric. With this notation and a bit of calculation the equation on the $V_m^{\pm}$ component becomes

$$2\mathbf{y}^T A_m^{\pm}\begin{bmatrix} 0 & 1 \\ -\delta & -c \end{bmatrix}\mathbf{y} = \mu_m^{\pm}\mathbf{y}^T A_m^{\pm}\mathbf{y} \mp \frac{p_0 q_2 y_1^2}{\mu_m^+ - \mu_m^-}.$$

Rearranging this equation we get an equation for $A_m^{\pm}$

$$\mathbf{y}^T A_m^{\pm}\begin{bmatrix} -\mu_m^{\pm} & 2 \\ -2c & -2\delta - \mu_m^{\pm} \end{bmatrix}\mathbf{y} = \mp\frac{p_0 q_2}{\mu_m^+ - \mu_m^-}\mathbf{y}^T\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\mathbf{y}.$$

Thus

$$A_m^{\pm} = \pm\frac{p_0 q_2}{\left(\mu_m^+ - \mu_m^-\right)\left(\mu_m^{\pm}\left(\mu_m^{\pm} + 2\delta\right) + 4c\right)}\begin{bmatrix} \mu_m^{\pm} + 2\delta & 1 \\ 1 & 0 \end{bmatrix}.$$

Hence we have determined the reduction map up to quadratic order in U^c. Now we are in a position to calculate the ordinary differential equation on $X_c \times \mathbb{R}^3$ up to cubic order.

Equation on $X_c \times \mathbb{R}^3$

From proposition 3.3.1 we have that the equation on $X_c \times \mathbb{R}^3$, which governs the dynamics on the centre manifold is

$$\begin{aligned} U^c_\tau &= \mathcal{B}U^c + \pi_c G\left(U^c + \psi\left(U^c, \alpha\right), \delta, c\right) \\ \alpha_\tau &= 0. \end{aligned}$$

Now if we set $\nu = 1$ and rearrange this equation we get,

$$\begin{aligned} U^c_\tau &= \begin{bmatrix} 0 & 1 \\ -\delta & -c \end{bmatrix} U^c + \pi_c H\left(U^c + \psi\left(U^c, \alpha\right)\right) \\ \delta_\tau &= 0 \\ c_\tau &= 0. \end{aligned}$$

Now if we write $U^c = \mathbf{y} = (y_1, y_2)$ and use the terms of the reduction map and J calculated in the previous subsection we can write the above equation as

$$\mathbf{y}_\tau = \begin{bmatrix} y_2 \\ -\delta y_1 - c y_2 - p_0 q_2 y_1^2 - p_0 q_3 y_1^3 - 2 q_2 y_1 \sum_{m \in \mathbb{Z}^2 \setminus \{0\}} p_{-m} \mathbf{y}^T\left(A_m^+ + A_m^-\right)\mathbf{y} \end{bmatrix} + \begin{bmatrix} 0 \\ O(\|\mathbf{y}\|^4) \end{bmatrix}.$$

Thus, since

$$\mathbf{y}^T A_m^{\pm} \mathbf{y} = \pm \frac{p_0 q_2}{\left(\mu_m^+ - \mu_m^-\right)\left(\mu_m^{\pm}\left(\mu_m^{\pm} + 2\delta\right) + 4c\right)} \left(\left(\mu_m^{\pm} + 2\delta\right) y_1^2 + 2 y_1 y_2\right),$$

we want to study the bifurcation behaviour of an equation of the form

$$\begin{aligned} \frac{dy_1}{d\tau} &= y_2 \\ \frac{dy_2}{d\tau} &= -\delta y_1 - c y_2 + \alpha y_1^2 + \beta y_1^3 + \gamma y_1^2 y_2 + O(\|\mathbf{y}\|^4), \end{aligned} \tag{3.9}$$

where α, β and $\gamma \in \mathbb{R}$. This equation is very similar to the type of equation which under goes a Bogdanov-Takens bifurcation, however we do not have the necessary quadratic terms for the standard Bogdanov-Takens bifurcation. But the cubic terms we include allow us to recover this type of behaviour.

The analysis of this bifurcation is based on the work in the books by Kuznetsov [35] and, by Guckenheimer and Holmes [21], on the Bogdanov-Takens bifurcation.

3.4.1 Bifurcations on the Centre Manifold

In this section we will study the bifurcations which occur on the local centre manifold to show that the bifurcation described in theorem 3.1.1 occur.

Thus we want to study the bifurcations close to $(0,0)$ as c and δ vary of the ordinary differential equation,

$$\begin{aligned}\frac{dy_1}{d\tau} &= y_2 \\ \frac{dy_2}{d\tau} &= -\delta y_1 - cy_2 + \alpha y_1^2 + \beta y_1^3 + \gamma y_1^2 y_2 + h(\mathbf{y}),\end{aligned} \tag{3.10}$$

where $h(\mathbf{y}) = O(\|\mathbf{y}\|^4)$. Throughout this section we will denote the vector field by,

$$K(\mathbf{y}) = \begin{bmatrix} y_2 \\ -\delta y_1 - cy_2 + \alpha y_1^2 + \beta y_1^3 + \gamma y_1^2 y_2 + h(\mathbf{y}) \end{bmatrix}.$$

We will start by proving the existence of a transcritical saddle-node bifurcation where two equilibria collide and exchange stability.

Transcritical saddle-node bifurcation

To prove the existence of a transcritical saddle-node bifurcation we need to show that a saddle and a node equilibria collide and exchange stability.

Thus we start by finding the equilibria of equation (3.10) close to $(0,0)$. A point $\mathbf{y} \in \mathbb{R}^2$ is a equilibria if and only if $y_2 = 0$ and $h(y_1, 0) + \beta y_1^3 + \alpha y_1^2 - \delta y_1 = 0$. Hence the point $(0,0)$ is an equilibrium for all $\delta \in \mathbb{R}$ and furthermore when $\delta = 0$ the value $y_1 = 0$ is a double root of the equation $h(y_1, 0) + \beta y_1^3 + \alpha y_1^2 = 0$. Thus if we define a function $E : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$E(y_1, \delta) = \frac{h(y_1, 0)}{y_1} + \beta y_1^2 + \alpha y_1 - \delta,$$

then $E(0,0) = 0$ and $D_{y_1} E(0,0) = \alpha \neq 0$. So by the implicit function theorem there exist open neighbourhoods of the origin $\Gamma_1, \Gamma_2 \subset \mathbb{R}$ and a function $\phi : \Gamma_1 \to \Gamma_2$, such that

$$E\left(\phi(\delta), \delta\right) = 0 \text{ for all } \delta \in \Gamma_1.$$

Furthermore by Taylor's theorem we have that

$$\phi(\delta) = \frac{\delta}{\alpha} + O(\delta^2).$$

Hence we have a second equilibria $(\phi(\delta), 0)$ which passes through $(0,0)$ when $\delta = 0$. For convenience we will suppress the argument of ϕ and denote the second equilibria by $(\phi, 0)$.

Thus to confirm that we have a transcritical saddle-node bifurcation when $\delta = 0$ and $c \neq 0$ it just remains to show that the two equilibria exchange stability. Hence we need to look at the eigenvalues of the linearizations about these two equilibria.

The linearization about $(0, 0)$ is

$$DK(0,0) = \begin{bmatrix} 0 & 1 \\ -\delta & -c \end{bmatrix},$$

which has eigenvalues,

$$\lambda^{\pm}_{\delta,c} = \frac{-c \pm \sqrt{c^2 - 4\delta}}{2}.$$

From these eigenvalues we see that if $c \neq 0$ then as δ changes from being positive to negative the equilibria changes from a stable or unstable equilibrium (depending on the sign of c) to a saddle equilibrium.

On the other hand the linearization about $(\phi, 0)$ is,

$$DK(\phi,0) = \begin{bmatrix} 0 & 1 \\ -\delta + 2\alpha\phi + 3\beta\phi^2 + D_{y_1}h(\phi,0) & -c + \gamma\phi^2 + D_{y_2}h(\phi,0) \end{bmatrix},$$

which has eigenvalues

$$\begin{aligned} \mu^{\pm}_{\delta,c} = & \frac{-\left(c - \gamma\phi^2 - D_{y_2}h(\phi,0)\right)}{2} \\ & \pm \frac{\sqrt{\left(c - \gamma\phi^2 - D_{y_2}h(\phi,0)\right)^2 - 4\left(\delta - 2\alpha\phi - 3\beta\phi^2 - D_{y_2}h(\phi,0)\right)}}{2}. \end{aligned}$$

Thus for $c \neq 0$ and δ sufficiently small

$$c - \gamma\phi^2 - D_{y_2}h(\phi,0) = c - \frac{\gamma\delta^2}{\alpha^2} + O(\delta^3) \neq 0,$$

and

$$\delta - 2\alpha\phi - 3\beta\phi^2 - D_{y_1}h(\phi,0) = -\delta + O(\delta^2).$$

So if $c \neq 0$ we have that as δ changes from being positive to negative the equilibrium change from a saddle equilibrium to a stable or unstable equilibria (depending on the sign of c).

Thus the non-zero equilibrium passes through the zero equilibrium and they exchange stability. Hence we have a transcritical saddle-node bifurcation occurring when $c \neq 0$ and $\delta = 0$.

Next we will prove the existence of Hopf bifurcations. Hence in the following two subsection we will find conditions on c and δ for which the two equilibria undergo Hopf bifurcations.

Hopf Bifurcation from $(0,0)$

We have the possibility of a Hopf bifurcation when the eigenvalues of the linearization are complex conjugates of each other with zero real part. For the equilibria $(0,0)$ this happens if $c = 0$ and $\delta > 0$, since in this case the eigenvalues of $DK(0,0)$ are

$$\lambda^{\pm}_{\delta,0} = \pm \mathrm{i}\sqrt{\delta}.$$

Now to prove that a Hopf bifurcation occurs when $c = 0$ and $\delta > 0$, we let $\delta > 0$ be fixed and use [35, Theorem 3.3 and Theorem 3.4]. In order to check the conditions of these theorems we need to calculate the first Lyapunov coefficient. We do this by writing our two dimensional real ordinary differential equation as a one dimensional complex differential equation in Poincaré normal form

$$z_\tau = \lambda(c)z + \sum_{2\leqslant l,k\leqslant 3} \frac{1}{l!k!} g_{k,l}(c) z^k \overline{z}^l + O(|z|^4),$$

where $\lambda(c) = \omega_1(c) + \mathrm{i}\omega_2(c)$. Once this is done the first Lyapunov coefficient is defined to be

$$l_1(0) = \frac{1}{2\omega_2(0)^2} \mathrm{Re}\left\{\mathrm{i} g_{20}(0) g_{11}(0) + \omega_2(0) g_{21}(0)\right\}.$$

Thus the first step to proving the existence of a Hopf bifurcation is to write equation (3.10) in Poincaré normal form. If $c^2 < 4\delta$ then the linearization of the vector field about $(0,0)$ has eigenvalues and eigenvectors,

$$\lambda^{\pm}_{\delta,c} = \frac{-c \pm \mathrm{i}\sqrt{4\delta - c^2}}{2} =: \omega_1(c) + \mathrm{i}\omega_2(c),$$

and

$$u_{\pm} = \begin{bmatrix} 1 \\ \lambda^{\pm}_{\delta,c} \end{bmatrix}.$$

Hence if we let $\mathbf{y} = zu_+ + \overline{z}u_-$, for $z \in \mathbb{C}$, equation (3.10) becomes

$$\begin{aligned} z_\tau u_+ + \overline{z}_\tau u_- = DK(0,0)\left(zu_+ + \overline{z}u_-\right) + \frac{\alpha(z+\overline{z})^2 + \beta(z+\overline{z})^3}{\lambda^+_{\delta,c} - \lambda^-_{\delta,c}}\left(u_+ - u_-\right) \\ + \frac{\gamma\left(z+\overline{z}\right)^2\left(\lambda^+_{\delta,c} z + \lambda^-_{\delta,c}\overline{z}\right)}{\lambda^+_{\delta,c} - \lambda^-_{\delta,c}}\left(u_+ - u_-\right) + O(|z|^4). \end{aligned}$$

Now if we look at the equation on the u_+ component we get an equation in terms of z,

$$z_\tau = \lambda^+_{\delta,c} z + \frac{\alpha(z+\overline{z})^2 + \beta(z+\overline{z})^3 + \gamma\,(z+\overline{z})^2\left(\lambda^+_{\delta,c} z + \lambda^-_{\delta,c}\overline{z}\right)}{\lambda^+_{\delta,c} - \lambda^-_{\delta,c}} + O(|z|^4).$$

The equation on the u_- component is just the complex conjugate of this equation. The above equation for z is in Poincaré normal form so we are now able to check the conditions of [35, Theorem 3.3 and Theorem 3.4].

Since

$$D_c \mathrm{Re}\left\{\lambda^+_{\delta,c}\right\} = -1 \neq 0$$

and

$$\begin{aligned} l_1(0) &= \frac{1}{2\omega_2(0)^2}\mathrm{Re}\left\{\mathrm{i}\left(\frac{2\alpha}{2\mathrm{i}\omega_2(0)}\right)\left(\frac{2\alpha}{2\mathrm{i}\omega_2(0)}\right) + 2\omega_2(0)\left(\frac{3\beta + \mathrm{i}\omega_2(0)\gamma}{2\mathrm{i}\omega_2(0)}\right)\right\} \\ &= \frac{1}{2\delta}\left(\frac{\delta\gamma}{\sqrt{\delta}}\right) \\ &= \frac{\gamma}{2\sqrt{\delta}}, \end{aligned}$$

by [35, Theorem 3.3 and Theorem 3.4] equation (3.10) has Hopf Bifurcation occurring at the $(0,0)$ equilibria when $\delta > 0$ and $c = 0$ provided $\gamma \neq 0$.

Hopf Bifurcation from $(\phi, 0)$

In this subsection we will find conditions on δ and c for which the equilibria $(\phi, 0)$ under goes a Hopf bifurcation.

We start by looking for values of δ and c where eigenvalues of the linearization of the vector field about $(\phi, 0)$ are complex conjugates of each other with zero real part.

The eigenvalues,

$$\begin{aligned} \mu^\pm_{\delta,c} = &-\frac{\left(c - \gamma\phi^2 - D_{y_2}h(\phi,0)\right)}{2} \\ &\pm \frac{\sqrt{\left(c - \gamma\phi^2 - D_{y_2}h(\phi,0)\right)^2 - 4\left(\delta - 2\alpha\phi - 3\beta\phi^2 - D_{y_1}h(\phi,0)\right)}}{2}, \end{aligned}$$

will have zero real part and be complex conjugates if and only if

$$c = \gamma\phi^2 + D_{y_2}h(\phi,0) = \frac{\gamma\delta^2}{\alpha^2} + O(\delta^3) =: c^H_\delta$$

and $\delta < 0$ is sufficiently small, since

$$-4(\delta - 2\alpha\phi - 3\beta\phi^2 - D_{y_1}h(\phi,0)) = 4(\delta + O(\delta^2)).$$

For these parameter values we have the possibility of a Hopf bifurcation.

To prove that a Hopf bifurcation occurs we will again use [35, Theorem 3.3 and Theorem 3.4]. Thus we need to perform a shift of co-ordinates to move the equilibria from $(\phi, 0)$ to $(0, 0)$ and then we need to write the two dimensional real ordinary differential equation produced as a one dimensional complex ordinary differential equation.

Therefore if we let,

$$\begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \end{bmatrix} = \begin{bmatrix} y_1 - \phi \\ y_2 \end{bmatrix},$$

then equation (3.10) becomes,

$$\begin{aligned} \frac{d\hat{y}_1}{d\tau} &= \hat{y}_2 \\ \frac{d\hat{y}_2}{d\tau} &= \left(-\delta + 2\alpha\phi + 3\beta\phi^2 + D_{y_1}h(\phi, 0)\right)\hat{y}_1 + \left(-c + \gamma\phi^2 + D_{y_2}h(\phi, 0)\right)\hat{y}_2 \\ &\quad + \left(\alpha + 3\beta\phi + D^2_{y_1}h\,(\phi, 0)\right)\hat{y}_1^2 + 2\left(\gamma\phi + D_{y_1}D_{y_2}h(\phi, 0)\right)\hat{y}_1\hat{y}_2 \\ &\quad + D^2_{y_2}h(\phi, 0)\hat{y}_2^2 + \left(\beta + D^3_{y_1}h(\phi, 0)\right)\hat{y}_1^3 + \left(\gamma + 3D^2_{y_1}D_{y_2}h(\phi, 0)\right)\hat{y}_1^2\hat{y}_2 \\ &\quad + 3D_{y_1}D^2_{y_2}h(\phi, 0)\hat{y}_1\hat{y}_2^2 + D^3_{y_2}h(\phi, 0)\hat{y}_2^3 + O(\|\hat{\mathbf{y}}\|^4) \end{aligned} \tag{3.11}$$

Now if $\left(c - \gamma\phi^2 - D_{y_2}h(\phi, 0)\right)^2 - 4\left(\delta - 2\alpha\phi - 3\beta\phi^2 - D_{y_1}h(\phi, 0)\right) < 0$ then $DK(\phi, 0)$ has eigenvalues and eigenvectors

$$\begin{aligned} \mu^{\pm}_{\delta,c} = &-\frac{\left(c - \gamma\phi^2 - D_{y_2}h(\phi, 0)\right)}{2} \\ &\pm \mathrm{i}\frac{\sqrt{4\left(\delta - 2\alpha\phi - 3\beta\phi^2 - D_{y_1}h(\phi, 0)\right) - \left(c - \gamma\phi^2 - D_{y_2}h(\phi, 0)\right)^2}}{2}, \end{aligned}$$

and

$$v_{\pm} = \begin{bmatrix} 1 \\ \mu^{\pm}_{\delta,c} \end{bmatrix}.$$

So if we let $\hat{\mathbf{y}} = zv_+ + \overline{z}v_-$ for $z \in \mathbb{C}$, then equation (3.11) becomes,

$$\begin{aligned}
z_\tau v_+ + \overline{z}_\tau v_- = DK(\phi, 0)\left(zv_+ + \overline{z}v_-\right) &+ \frac{\left(\alpha + 3\beta\phi + D_{y_1}^2 h\left(\phi, 0\right)\right)\left(z + \overline{z}\right)^2}{\mu_{\delta,c}^+ - \mu_{\delta,c}^-}\left(v_+ - v_-\right) \\
&+ \frac{\left(2\gamma\phi + 2D_{y_1}D_{y_2}h(\phi, 0)\right)\left(z + \overline{z}\right)\left(\mu_{\delta,c}^+ z + \mu_{\delta,c}^- \overline{z}\right)}{\mu_{\delta,c}^+ - \mu_{\delta,c}^-}\left(v_+ - v_-\right) \\
&+ \frac{D_{y_2}^2 h(\phi, 0)(\mu_{\delta,c}^+ z + \mu_{\delta,c}^- \overline{z})^2 + \left(\beta + D_{y_1}^3 h(\phi, 0)\right)\left(z + \overline{z}\right)^3}{\mu_{\delta,c}^+ - \mu_{\delta,c}^-}\left(v_+ - v_-\right) \\
&+ \frac{\left(\gamma + 3D_{y_1}^2 D_{y_2} h(\phi, 0)\right)\left(z + \overline{z}\right)^2(\mu_{\delta,c}^+ z + \mu_{\delta,c}^- \overline{z})}{\mu_{\delta,c}^+ - \mu_{\delta,c}^-}\left(v_+ - v_-\right) \\
&+ \frac{3D_{y_1}D_{y_2}^2 h(\phi, 0)(z + \overline{z})(\mu_{\delta,c}^+ z + \mu_{\delta,c}^- \overline{z})^2}{\mu_{\delta,c}^+ - \mu_{\delta,c}^-}\left(v_+ - v_-\right) \\
&+ \frac{D_{y_2}^3 h(\phi, 0)(\mu_{\delta,c}^+ z + \mu_{\delta,c}^- \overline{z})^3}{\mu_{\delta,c}^+ - \mu_{\delta,c}^-}\left(v_+ - v_-\right) + O(|z|^4)
\end{aligned}$$

and if we look at the equation on the v_+ component we get an equation in terms of z

$$\begin{aligned}
z_\tau = \mu_{\delta,c}^+ z &+ \frac{\left(\alpha + 3\beta\phi + D_{y_1}^2 h\left(\phi, 0\right)\right)\left(z + \overline{z}\right)^2}{\mu_{\delta,c}^+ - \mu_{\delta,c}^-} \\
&+ \frac{\left(2\gamma\phi + 2D_{y_1}D_{y_2}h(\phi, 0)\right)\left(z + \overline{z}\right)\left(\mu_{\delta,c}^+ z + \mu_{\delta,c}^- \overline{z}\right) + D_{y_2}^2 h(\phi, 0)(\mu_{\delta,c}^+ z + \mu_{\delta,c}^- \overline{z})^2}{\mu_{\delta,c}^+ - \mu_{\delta,c}^-} \\
&+ \frac{\left(\beta + D_{y_1}^3 h(\phi, 0)\right)\left(z + \overline{z}\right)^3 + \left(\gamma + 3D_{y_1}^2 D_{y_2} h(\phi, 0)\right)\left(z + \overline{z}\right)^2(\mu_{\delta,c}^+ z + \mu_{\delta,c}^- \overline{z})}{\mu_{\delta,c}^+ - \mu_{\delta,c}^-} \\
&+ \frac{3D_{y_1}D_{y_2}^2 h(\phi, 0)(z + \overline{z})(\mu_{\delta,c}^+ z + \mu_{\delta,c}^- \overline{z})^2 + D_{y_2}^3 h(\phi, 0)(\mu_{\delta,c}^+ z + \mu_{\delta,c}^- \overline{z})^3}{\mu_{\delta,c}^+ - \mu_{\delta,c}^-} \\
&+ O(|z|^4).
\end{aligned}$$

Hence we have a complex ordinary differential equation in Poincaré normal form, so we are in a position to check the conditions for a Hopf Bifurcation to occur. Thus if we let

$$\hat{\omega} = \sqrt{\delta - 2\alpha\phi - 3\beta\phi^2 - D_{y_1}h(\phi, 0)} = \left|\mu_{\delta,c_\delta^H}^+\right|,$$

then,

$$D_c \mathrm{Re}\left\{\mu_{\delta,c_\delta^H}^+\right\} = -1 \neq 0,$$

and we have,

$$
\begin{aligned}
&l^1(c_\delta^H) \\
&= \frac{1}{2\hat\omega}\mathrm{Re}\Bigg\{\mathrm{i}\left(2\frac{\left(\alpha+3\beta\phi+D_{y_1}^2h\left(\phi,0\right)\right)+2\mathrm{i}\hat\omega\left(\gamma\phi+D_{y_1}D_{y_2}h(\phi,0)\right)+(\mathrm{i}\hat\omega)^2D_{y_2}^2h(\phi,0)}{2\mathrm{i}\hat\omega}\right) \\
&\times\left(\frac{2\left(\alpha+3\beta\phi+D_{y_1}^2h\left(\phi,0\right)\right)+(\mathrm{i}\hat\omega-\mathrm{i}\hat\omega)\left(2\gamma\phi+2D_{y_1}D_{y_2}h(\phi,0)\right)-2(\mathrm{i}\hat\omega)^2D_{y_2}^2h(\phi,0)}{2\mathrm{i}\hat\omega}\right) \\
&+2\hat\omega\Bigg(\frac{\left(\beta+D_{y_1}^3h(\phi,0)\right)+(2\mathrm{i}\hat\omega-\mathrm{i}\hat\omega)\left(\gamma+3D_{y_1}^2D_{y_2}h(\phi,0)\right)}{2\mathrm{i}\hat\omega} \\
&\qquad+\frac{3((\mathrm{i}\hat\omega)^2-2(\mathrm{i}\hat\omega)^2)D_{y_1}D_{y_2}^2h(\phi,0)-(\mathrm{i}\hat\omega)^3D_{y_2}^3h(\phi,0)}{2\mathrm{i}\hat\omega}\Bigg)\Bigg\} \\
&= \frac{1}{2\hat\omega^2}\Bigg(\frac{2\left(\gamma\phi+D_{y_1}D_{y_2}h(\phi,0)\right)\left(\alpha+3\beta\phi+D_{y_1}^2h(\phi,0)+\hat\omega^2D_{y_2}^2h(\phi,0)\right)}{\hat\omega} \\
&\qquad+\hat\omega\left(\gamma+3D_{y_1}^2D_{y_2}h(\phi,0)+\hat\omega^3D_{y_2}^3h(\phi,0)\right)\Bigg).
\end{aligned}
$$

Now, since

$$\phi = \frac{\delta}{\alpha} + O(\delta^2)$$

and

$$\hat\omega = \sqrt{|\delta| + O(\delta^2)},$$

it follows with a bit of calculation that

$$
\begin{aligned}
l^1(c_\delta^H) &= \frac{1}{\hat\omega^3}\left(\left(2\gamma\delta+O(\delta^2)\right)+\hat\omega^2\left(\gamma+O(\delta)\right)\right) \\
&= -\frac{1}{2\hat\omega^3}\left(\gamma\left|\delta\right|+O(\delta^2)\right)\neq 0,
\end{aligned}
$$

for sufficiently small δ.

Hence by [35, Theorem 3.3 and Theorem 3.4] we have a Hopf bifurcation occurring at the equilibria $(\phi, 0)$ when $\delta < 0$ is sufficiently small and,

$$c = \gamma\phi^2 + D_{y_2}h(\phi,0) = \frac{\gamma\delta^2}{\alpha^2} + O(\delta^3),$$

provided $\gamma \neq 0$.

We have now proved the existence of two types of bifurcation described in theorem 3.1.1 and we have the bifurcation diagram shown in figure 3-2.

To complete the bifurcation picture from theorem 3.1.1 all that remains is to prove the existence of the saddle-homoclinic bifurcations.

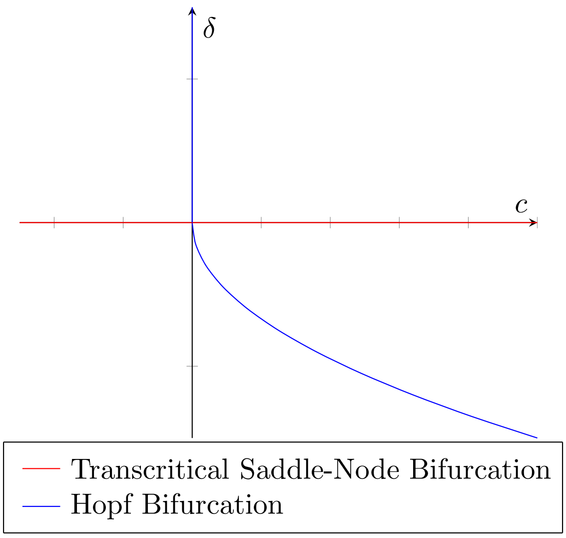


Figure 3-2: *Bifurcation diagrams for Hopf and Transcritical saddle-node bifurcations for* $\gamma > 0$.

In order to find the points where the saddle-homoclinic bifurcations occur we need to find values of δ and c where the saddle equilibrium has a homoclinic orbit. We do this by performing a blow-up rescaling on equation (3.10) and then using a Melnikov functional argument.

Saddle-Homoclinic Bifurcation for $\delta > 0$

Let $\rho > 0$ then under the blow-up rescaling

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \sqrt{\rho} \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix},$$

equation (3.10) becomes

$$\begin{aligned} \frac{d\tilde{y}_1}{d\tau} &= \tilde{y}_2 \\ \frac{d\tilde{y}_2}{d\tau} &= -\delta\tilde{y}_1 + \tilde{\alpha}\tilde{y}_1^2 + \rho\left(-\tilde{c}\tilde{y}_2 + \beta\tilde{y}_1^3 + \gamma\tilde{y}_1^2\tilde{y}_2\right) + \rho^{\frac{3}{2}}\tilde{h}(\tilde{\mathbf{y}}, \rho), \end{aligned} \tag{3.12}$$

where $\tilde{\alpha} = \sqrt{\rho}\alpha$, $\tilde{c} = c/\rho$ and $\tilde{h}(\tilde{\mathbf{y}}, \rho) = h(\sqrt{\rho}\tilde{\mathbf{y}})/\rho^2$.

Now if we set $\rho = 0$ in equation (3.12) then we get the Hamiltonian system

$$\begin{aligned} \frac{d\tilde{y}_1}{d\tau} &= \tilde{y}_2 \\ \frac{d\tilde{y}_2}{d\tau} &= -\delta\tilde{y}_1 + \tilde{\alpha}\tilde{y}_1^2, \end{aligned} \tag{3.13}$$

which has Hamiltonian

$$H(\tilde{y}_1, \tilde{y}_2) = \frac{\tilde{y}_2^2}{2} + \frac{\delta \tilde{y}_1^2}{2} - \frac{\tilde{\alpha}\tilde{y}_1^3}{3},$$

and a homoclinic orbit

$$\tilde{y}_{1,0}(\tau) = \frac{\delta}{\hat{\alpha}}\left(1 - \frac{3}{2}\mathrm{sech}^2\left(\frac{\sqrt{\delta}}{2}\tau\right)\right)$$
$$\tilde{y}_{2,0}(\tau) = \frac{3\delta^{\frac{3}{2}}}{2\hat{\alpha}}\mathrm{sech}^2\left(\frac{\sqrt{\delta}}{2}\tau\right)\tanh\left(\frac{\sqrt{\delta}}{2}\tau\right).$$

Therefore we can view equation (3.12) as a perturbation of the Hamiltonian system (3.13). Thus we want to find the values of $\tilde{c}$ and δ for which homoclinic orbit persists under this perturbation. We will do this by using a Melnikov functional argument.

Now equation (3.12) has a saddle equilibrium at $(\rho^{-\frac{1}{2}}\phi, 0)$ so if we define the separation function

$$Sep : \mathbb{R} \to \mathbb{R},$$

in the same way as [13, Section 6.1], then this function gives an approximation to the separation of the unstable and stable manifolds of this saddle equilibrium. Furthermore it has the important property that $Sep(\rho) = 0$ if and only if the unstable and stable manifolds intersect. Thus if $Sep(\rho) = 0$ then equation (3.12) has a homoclinic orbit.

Thus, since from [13, Proposition 6.2] we know that

$$\frac{d}{d\rho}Sep(0) = \mathcal{M}(\delta, \tilde{c}),$$

where

$$\mathcal{M}(\delta, \tilde{c}) = \int_{-\infty}^{\infty} \hat{y}_{2,0}(\tau)\left(-\tilde{c}\tilde{y}_{2,0}(\tau) + \beta\tilde{y}_{1,0}(\tau)^3 + \gamma\tilde{y}_{1,0}(\tau)^2\tilde{y}_{2,0}(\tau)\right)\mathrm{d}\tau$$

is the Melnikov functional, and $Sep(0) = 0$ it follows that for $\rho > 0$ small,

$$Sep(\rho) = \rho\mathcal{M}(\delta, \tilde{c}) + o(\rho).$$

Therefore if we can find values of δ and $\tilde{c}$ for which $\mathcal{M}(\delta, \tilde{c})$ changes sign then for $\rho > 0$ sufficiently small there will exist values of δ and $\tilde{c}$ close to these values such that $Sep(\rho) = 0$. Hence we look for values of δ and $\tilde{c}$ such that

$$\mathcal{M}(\delta, \tilde{c}) = 0 \text{ and } D_{\tilde{c}}\mathcal{M}(\delta, \tilde{c}) \neq 0.$$

Now

$$\mathcal{M}(\delta, \tilde{c}) = 0 \Leftrightarrow \int_{-\infty}^{\infty} \hat{y}_{2,0}(\tau)\left(-\tilde{c}\tilde{y}_{2,0}(\tau) + \beta\tilde{y}_{1,0}(\tau)^3 + \gamma\tilde{y}_{1,0}(\tau)^2\tilde{y}_{2,0}(\tau)\right)\mathrm{d}\tau = 0,$$

so since

$$\tilde{y}_{1,0}(\tau)^3\tilde{y}_{2,0}(\tau) = \frac{3\delta^{\frac{9}{2}}}{2\tilde{\alpha}^4}\left(1 - \frac{3}{2}\text{sech}^2\left(\frac{\sqrt{\delta}}{2}\tau\right)\right)^3 \text{sech}^2\left(\frac{\sqrt{\delta}}{2}\tau\right)\tanh\left(\frac{\sqrt{\delta}}{2}\tau\right)$$

is an odd function and

$$\tilde{y}_{2,0}(\tau)^2 = \frac{9\delta^3}{4\tilde{\alpha}^2}\text{sech}^4\left(\frac{\sqrt{\delta}}{2}\tau\right)\tanh^2\left(\frac{\sqrt{\delta}}{2}\tau\right) \geqslant 0$$

is even, continuous and not zero everywhere, it follows that

$$\begin{aligned}\mathcal{M}(\delta,\tilde{c}) = 0 \Leftrightarrow \tilde{c} &= \frac{\int_{-\infty}^{\infty}\gamma\tilde{y}_{1,0}(\tau)^2\tilde{y}_{2,0}(\tau)^2\mathrm{d}\tau}{\int_{-\infty}^{\infty}\gamma\tilde{y}_{2,0}(\tau)^2\mathrm{d}\tau} \\ &= \frac{\gamma\delta^2\int_{-\infty}^{\infty}\left(1 - \frac{3}{2}\text{sech}^2\left(\frac{\sqrt{\delta}}{2}\tau\right)\right)^2\text{sech}^4\left(\frac{\sqrt{\delta}}{2}\tau\right)\tanh^2\left(\frac{\sqrt{\delta}}{2}\tau\right)\mathrm{d}\tau}{\tilde{\alpha}^2\int_{-\infty}^{\infty}\text{sech}^4\left(\frac{\sqrt{\delta}}{2}\tau\right)\tanh^2\left(\frac{\sqrt{\delta}}{2}\tau\right)\mathrm{d}\tau}.\end{aligned}$$

So with a bit of calculation using the identity

$$\text{sech}^2\left(\frac{\sqrt{\delta}}{2}\tau\right) = 1 - \tanh^2\left(\frac{\sqrt{\delta}}{2}\tau\right),$$

and the fact that for $k \in \mathbb{N}$ even,

$$\int_{-\infty}^{\infty}\text{sech}^2(\sigma)\tanh^k(\sigma)\mathrm{d}\sigma = \frac{2}{k+1},$$

we get

$$\mathcal{M}(\delta,\tilde{c}) = 0 \Leftrightarrow \tilde{c} = \frac{\gamma\delta^2}{7\tilde{\alpha}^2}.$$

Also we have that

$$D_{\tilde{c}}\mathcal{M}(\delta,\tilde{c}) = -\int_{-\infty}^{\infty}\tilde{y}_{2,0}^2(\tau)\mathrm{d}\tau \neq 0,$$

so the Melnikov function changes sign at $\tilde{c} = \gamma\delta^2/7\tilde{\alpha}^2$.

Now if we fix $\delta > 0$ and let

$$\tilde{c}_\delta^\pm = \frac{\gamma\delta^2}{7\tilde{\alpha}^2} \pm \frac{\delta^3}{\tilde{\alpha}^2},$$

then

$$\begin{aligned}\mathcal{M}(\delta, \tilde{c}_\delta^{\pm}) &= \mp \int_{-\infty}^{\infty} \frac{9\delta^6}{4\tilde{\alpha}^4} \operatorname{sech}^4\left(\frac{\sqrt{\delta}}{2}\tau\right)\tanh^2\left(\frac{\sqrt{\delta}}{2}\tau\right) \mathrm{d}\tau \\ &= \mp \frac{9\delta^{\frac{11}{2}}}{2\tilde{\alpha}^4} \int_{-\infty}^{\infty} \operatorname{sech}^4(\sigma)\tanh^2(\sigma)\mathrm{d}\sigma,\end{aligned}$$

so for $\rho > 0$ sufficiently small

$$Sep(\rho) = \rho\mathcal{M}(\delta, \tilde{c}) + o(\rho)$$

will be negative if $\tilde{c} = \tilde{c}_\delta^+$ and positive if $\tilde{c} = \tilde{c}_\delta^-$. So there will exist a $\tilde{c}_\star$ between $\tilde{c}_\delta^+$ and $\tilde{c}_\delta^-$ such that $Sep(\rho) = 0$ and because of the way we choose $\tilde{c}_\delta^+$ and $\tilde{c}_\delta^-$ we have that

$$\frac{\gamma\delta^2}{7\tilde{\alpha}^2} - \frac{\delta^3}{\tilde{\alpha}^2} \leqslant \tilde{c}_\star \leqslant \frac{\gamma\delta^2}{7\tilde{\alpha}^2} + \frac{\delta^3}{\tilde{\alpha}^2}.$$

Hence we have found a homoclinic orbit of the blown-up equation (3.12).

Now if we undo the blow-up rescaling we find that the original equation (3.10) has a homoclinic orbit if $c = c_\star$ where

$$\frac{\gamma\delta^2}{7\alpha^2} - \frac{\delta^3}{\alpha^2} \leqslant c_\star := \rho\tilde{c}_\star \leqslant \frac{\gamma\delta^2}{7\alpha^2} + \frac{\delta^3}{\alpha^2},$$

since $\tilde{\alpha} = \sqrt{\rho}\alpha$ and $\tilde{c} = c/\rho$. Thus we have found where a saddle-homoclinic bifurcation occurs for $\delta > 0$.

Saddle-Homoclinic Bifurcation for $\delta < 0$

Let $\rho > 0$ then under the same blow-up rescaling as the previous section

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \sqrt{\rho} \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix},$$

equation (3.10) becomes

$$\begin{aligned}\frac{d\tilde{y}_1}{d\tau} &= \tilde{y}_2 \\ \frac{d\tilde{y}_2}{d\tau} &= |\delta|\,\tilde{y}_1 + \tilde{\alpha}\tilde{y}_1^2 + \rho\left(-\tilde{c}\tilde{y}_2 + \beta\tilde{y}_1^3 + \gamma\tilde{y}_1^2\tilde{y}_2\right) + \rho^{\frac{3}{2}}\tilde{h}(\tilde{\mathbf{y}}, \rho),\end{aligned} \tag{3.14}$$

where $\tilde{\alpha} = \sqrt{\rho}\alpha$, $\tilde{c} = c/\rho$ and $\tilde{h}(\tilde{\mathbf{y}}, \rho) = h(\sqrt{\rho}\tilde{\mathbf{y}})/\rho^2$.

If we set $\rho = 0$ then this equation becomes the Hamiltonian system

$$\begin{aligned}\frac{d\tilde{y}_1}{d\tau} &= \tilde{y}_2 \\ \frac{d\tilde{y}_2}{d\tau} &= |\delta|\,\tilde{y}_1 + \tilde{\alpha}\tilde{y}_1^2,\end{aligned} \tag{3.15}$$

which has Hamiltonian

$$H(\tilde{y}_1, \tilde{y}_2) = \frac{\tilde{y}_2^2}{2} - \frac{|\delta|\,\tilde{y}_1^2}{2} - \frac{\tilde{\alpha}\tilde{y}^3}{3},$$

and homoclinic orbit

$$\begin{aligned}\tilde{y}_{1,1}(\tau) &= -\frac{3\,|\delta|}{2\tilde{\alpha}}\mathrm{sech}^2\left(\frac{\sqrt{|\delta|}}{2}\tau\right) \\ \tilde{y}_{2,1}(\tau) &= \frac{3\,|\delta|^{\frac{3}{2}}}{2\tilde{\alpha}}\mathrm{sech}^2\left(\frac{\sqrt{|\delta|}}{2}\tau\right)\tanh\left(\frac{\sqrt{|\delta|}}{2}\tau\right).\end{aligned}$$

Thus we can view equation (3.14) as a perturbation of the Hamiltonian system (3.15). Hence we want to find values of $\tilde{c}$ and δ such that the homoclinic orbit persists under this perturbation.

Using an identical argument to the one given in the previous subsection we can estimate the splitting of the stable and unstable manifolds of the saddle equilibria at $(0, 0)$ for $\rho > 0$ sufficiently small by the separation function,

$$Sep(\rho) = \rho\mathcal{M}(\delta, \tilde{c}) + o(\rho).$$

In this case the Melnikov functional is

$$\mathcal{M}(\delta, \tilde{c}) = \int_{-\infty}^{\infty} \hat{y}_{2,1}(\tau)\left(-\tilde{c}\tilde{y}_{2,1}(\tau) + \beta\tilde{y}_{1,1}(\tau)^3 + \gamma\tilde{y}_{1,1}(\tau)^2\tilde{y}_{2,1}(\tau)\right)\mathrm{d}\tau.$$

Now similar calculations to the previous section show that the Melnikov functional $\mathcal{M}(\delta, \tilde{c})$ changes sign when

$$\tilde{c} = \frac{6\gamma\delta^2}{7\tilde{\alpha}}.$$

Thus for $\delta < 0$ fixed and $\rho > 0$ sufficiently small there exist a $\tilde{c}^\star$ such that

$$\frac{6\gamma\delta^2}{7\tilde{\alpha}^2} - \frac{|\delta|^3}{\tilde{\alpha}^2} \leqslant \tilde{c}^\star \leqslant \frac{6\gamma\delta^2}{7\tilde{\alpha}^2} + \frac{|\delta|^3}{\tilde{\alpha}^2},$$

and if $c = \tilde{c}^\star$ then $Sep(\rho) = 0$ and equation (3.14) has a homoclinic orbit. Hence if we undo the blow-up rescaling we find that equation (3.10) has a homoclinic orbit for $c = c^\star$ where

$$\frac{6\gamma\delta^2}{7\alpha^2} - \frac{|\delta|^3}{\alpha^2} \leqslant c^\star := \rho\tilde{c}^\star \leqslant \frac{6\gamma\delta^2}{7\alpha^2} + \frac{|\delta|^3}{\alpha^2}.$$

Thus we have found all the bifurcations described in theorem 3.1.1 and we have the bifurcation diagram shown in figure 3-3 provided $\gamma \neq 0$. Thus we get a condition on the periodic function p from the nonlinearity, since $\gamma \neq 0$ if and only if

$$\sum_{m \in \mathbb{Z}^2 \setminus \{0\}} \frac{p_{-m}}{\mu_m^+ - \mu_m^-} \left(\frac{1}{\mu_m^+ \left(\mu_m^+ + 2\delta\right) + 4c} + \frac{1}{\mu_m^- \left(\mu_m^- + 2\delta\right) + 4c} \right) \neq 0.$$

This condition is satisfied for a non-empty open set of $p \in H^2\left(T^2\right)$. Hence we have proved the first part of theorem 3.1.1.

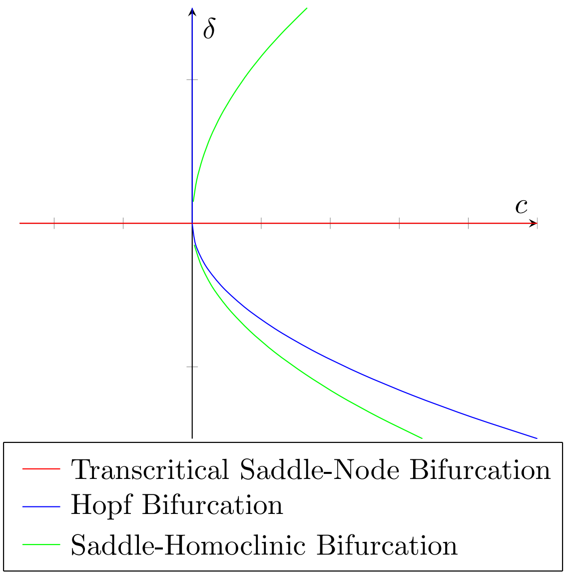


Figure 3-3: *Bifurcation diagram for transcritical saddle-node, Hopf and saddle-homoclinic bifurcations for $\gamma > 0$.*

In the next section we will look at the solutions on the centre manifold that are created by these bifurcations to complete the proof of theorem 3.1.1.

3.4.2 Solutions on the Centre Manifold

In this section we will discuss the solution created on the centre manifold by the bifurcation described in the previous section. Since we have a local centre manifold about the origin, solution on the centre manifold are only solution of the spatial dynamical system if they stay within the open neighbourhood Ω of the origin from proposition 3.3.1.

Thus to ensure that the solutions created by these bifurcations are solutions of the spatial dynamical system (3.3) we need to ensure that when they are mapped onto the local centre manifold they stay within Ω.

From lemma 3.3.10 we know that if we choose ε sufficiently small then Ω will be large enough that we can set $\nu = 1$ to eliminate the perturbation.

Now to show that a bounded solution U^c created by one of the bifurcations lies on the local centre manifold we need to show that

$$(U^c(\tau) + \psi\left(U^c(\tau), \alpha\right), \alpha) \in \Omega \text{ for all } \tau \in \mathbb{R}.$$

In the next three subsections we will discuss whether each of the bifurcation we have found create solutions which lie on the centre manifold.

Heteroclinic Connection Type Solutions

If we fix $c \neq 0$ then, by theorem 2.1.1, for $\delta \in \mathbb{R}$ with $|\delta| > 0$ sufficiently small there exists a generalised travelling wave solution corresponding to a heteroclinic connection. Thus generalised travelling wave solutions with a heteroclinic structure in τ are created by the transcritical saddle node bifurcation which occurs when $c \neq 0$ and $\delta = 0$. So we have found the first type of solution described in theorem 3.1.1.

Periodic Type Solutions

If $\delta > 0$ is fixed then we have a Hopf bifurcation occurring at the $(0, 0)$ equilibria when $c = 0$. This bifurcation creates a periodic orbit $U^c_p(\tau)$ which surrounds the equilibria as c becomes negative or positive depending on the sign of γ. Then as $|c|$ increases the size of this periodic orbit grows. Thus we have that

$$\sup_{t\in\mathbb{R}} \left\|U^c_p(\tau)\right\|_{X_c} \to 0 \text{ as } |c| \to 0.$$

Hence, since $\psi(0, (\delta, 0, 1)) = 0$ and ψ is continuous it follows that for $|c|$ and δ sufficiently small

$$\left(U^c_p(\tau) + \psi\left(U^c_p(\tau), \alpha\right), \alpha\right) \in \Omega \text{ for all } \tau \in \mathbb{R}.$$

So for $\delta > 0$ and $|c|$ sufficiently small we have a periodic orbit on the centre manifold.

On the other hand if $\delta < 0$ then we have a Hopf bifurcation occurring at the $(\phi, 0)$ equilibria when

$$c = c^H_\delta = \frac{\gamma\delta^2}{\alpha^2} + O(\delta^3).$$

This bifurcation creates a periodic orbit $\tilde{U}^c_p(\tau)$ which surrounds the equilibrium $(\phi, 0)$ for $c > c^H_\delta$ or $c < c^H_\delta$ depending on the sign of γ. This periodic orbit will be such that,

$$\sup_{t\in\mathbb{R}} \left\|\tilde{U}^c_p(\tau)\right\|_{X_c} \to \|(\phi, 0)\|_{X_c} \text{ as } c \to c^H_\delta.$$

Thus, since

$$\|(\phi, 0)\|_{X_c} \to 0 \text{ as } \delta \to 0,$$

$\psi(0, (\delta, c, 1)) = 0$ and ψ is continuous, it follow that for c sufficiently close to c_H^δ and δ sufficiently small

$$\left(\tilde{U}_p^c(\tau) + \psi\left(\tilde{U}_p^c(\tau), \alpha\right), \alpha\right) \in \Omega \text{ for all } \tau \in \mathbb{R}.$$

Hence near the values of c and δ where a Hopf bifurcation occurs we have a periodic orbit in τ on the local centre manifold. This periodic orbit will then correspond to a generalised traveling wave profile function v with the property

$$v(\tau, \xi) = v(\tau + T, \xi) \text{ for some } T > 0.$$

Thus we have found the second type of solution described in theorem 3.1.1.

Homoclinic Type Solutions

If $\delta > 0$ then for

$$c = \frac{\gamma\delta^2}{7\alpha^2} + O(\delta^3),$$

we have the existence of a saddle-homoclinic bifurcation which creates a homoclinic orbit $U_{Hom}^c(\tau)$. This homoclinic orbit is a bounded solution and thus to show that it lies on the local centre manifold, we just need to show that

$$\sup_{\tau\in\mathbb{R}} \|U_{Hom}^c(\tau)\|_{X_c} \to 0 \text{ as } \delta \to 0,$$

since $\psi\left(0, (\delta, c, 1)\right) = 0$ and ψ is continuous.

In order to show this result we need to look at how we proved the existence of a homoclinic orbit to find the saddle-homoclinic bifurcation. To prove the existence of a homoclinic orbit we performed a blow-up rescaling on equation (3.10) to get the equation

$$\begin{aligned}\frac{d\tilde{y}_1}{d\tau} &= \tilde{y}_2\\ \frac{d\tilde{y}_2}{d\tau} &= -\delta\tilde{y}_1 + \tilde{\alpha}\tilde{y}_1^2 + \rho\left(-\tilde{c}\tilde{y}_2 + \beta\tilde{y}_1^3 + \gamma\tilde{y}_1^2\tilde{y}_2\right) + \rho^{\frac{3}{2}}\tilde{h}(\tilde{\mathbf{y}}, \rho),\end{aligned}$$

which we viewed as a perturbation of the Hamiltonian system

$$\begin{aligned}\frac{d\tilde{y}_1}{d\tau} &= \tilde{y}_2\\ \frac{d\tilde{y}_2}{d\tau} &= -\delta\tilde{y}_1 + \tilde{\alpha}\tilde{y}_1^2.\end{aligned}$$

This Hamiltonian system has a homoclinic orbit

$$\tilde{y}_{1,0}(\tau) = \frac{\delta}{\hat{\alpha}}\left(1 - \frac{3}{2}\text{sech}^2\left(\frac{\sqrt{\delta}}{2}\tau\right)\right)$$
$$\tilde{y}_{2,0}(\tau) = \frac{3\delta^{\frac{3}{2}}}{2\hat{\alpha}}\text{sech}^2\left(\frac{\sqrt{\delta}}{2}\tau\right)\tanh\left(\frac{\sqrt{\delta}}{2}\tau\right).$$

and we were able to show that for ρ sufficiently small the perturbed equation has a homoclinic orbit when, $\tilde{c} = \tilde{c}_\star$.

From Melnikov theory we know that the stable and unstable manifolds stay within order $O(\rho)$ of the unperturbed stable and unstable manifolds, so it follows that the homoclinic orbit of the perturbed system

$$\tilde{y}^p_{1,0}(\tau) = \tilde{y}_{1,0}(\tau) + O(\rho)$$
$$\tilde{y}^p_{2,0}(\tau) = \tilde{y}_{2,0}(\tau) + O(\rho).$$

Thus if we undo the blow-up rescaling we get that the homoclinic orbit for the original equation,

$$U^c_{Hom}(\tau) = \begin{bmatrix} \sqrt{\rho}\tilde{y}_{1,0}(\tau) + O(\rho^{\frac{3}{2}}) \\ \sqrt{\rho}\tilde{y}_{2,0}(\tau) + O(\rho^{\frac{3}{2}}) \end{bmatrix} = \begin{bmatrix} \frac{\delta}{\alpha}\left(1 - \frac{3}{2}\text{sech}^2\left(\frac{\sqrt{\delta}}{2}\tau\right)\right) + O(\rho^{\frac{3}{2}}) \\ \frac{3\delta^{\frac{3}{2}}}{2\hat{\alpha}}\text{sech}^2\left(\frac{\sqrt{\delta}}{2}\tau\right)\tanh\left(\frac{\sqrt{\delta}}{2}\tau\right). + O(\rho^{\frac{3}{2}}) \end{bmatrix}$$

Now, since we have to choose $\rho > 0$ sufficiently small, we can always choose $\rho < \delta$ to get

$$U^c_{Hom}(\tau) = \begin{bmatrix} \frac{\delta}{\alpha}\left(1 - \frac{3}{2}\text{sech}^2\left(\frac{\sqrt{\delta}}{2}\tau\right)\right) + O(\delta^{\frac{3}{2}}) \\ \frac{3\delta^{\frac{3}{2}}}{2\alpha}\text{sech}^2\left(\frac{\sqrt{\delta}}{2}\tau\right)\tanh\left(\frac{\sqrt{\delta}}{2}\tau\right). + O(\delta^{\frac{3}{2}}) \end{bmatrix}.$$

Thus it follows that

$$\sup_{t\in\mathbb{R}} \left\|\tilde{U}^c_{Hom}(\tau)\right\|_{X_c} \to 0 \text{ as } \delta \to 0,$$

and for δ sufficiently small we have homoclinic orbit in τ on the local centre manifold. This homoclinic orbit will then correspond to a generalised travelling wave solution with the property

$$v(\tau, \xi) \to v_0(\xi) \text{ as } \tau \to \pm\infty.$$

Thus we have found the third type of solution described in theorem 3.1.1. For $\delta < 0$ an almost identical argument will show that the for δ sufficiently small the saddle-homoclinic solution will generate a homoclinic orbit on the local centre manifold and thus a generalised travelling wave with homoclinic structure.

Thus we have shown that for sufficiently small δ and c the bifurcation create the three types of solutions that are described in theorem 3.1.1. Hence we have completed the proof of theorem 3.1.1.

□

3.5 Conclusion

In this chapter we have considered a reaction diffusion equation of the form

$$u_t = \operatorname{div}(A\nabla u) + \delta u + p\left(\frac{x}{\varepsilon}\right) q(u),$$

and we have shown that there exists three different kinds of travelling wave solution depending on the values of c and δ provided that $\int_{T^2} p(s)\mathrm{d}s \neq 0$ and, $|\delta|$, $|c|$ and $\varepsilon > 0$ are sufficiently small.

This result allows us to see that the generalised travelling wave solutions we found in chapter two are created by a transcritical saddle-node bifurcation and that the nature of the travelling wave solution changes as c and δ vary around zero. In particular we have shown that generalised travelling wave solution can exist when c small they may just have a different structure to those that exist when $|\delta|$ is small.

3.6 Technical Result

In this section we present a more detailed version of the result described in theorem 3.1.1.

Consider a reaction diffusion equation in $\mathbb{R}^2$ with a nonlinearity which is periodic in space,

$$u_t = \operatorname{div}(A\nabla u) + f\left(\frac{x}{\varepsilon}, u\right),$$

where A is a real symmetric matrix, $\varepsilon > 0$ and f is the nonlinearity. We assume that the nonlinearity f is of the form,

$$f\left(\frac{x}{\varepsilon}, u\right) = \delta u + p\left(\frac{x}{\varepsilon}\right) q(u),$$

where $\delta \in \mathbb{R}$, $p \in H^2\left(T^2\right)$ and, $q \in C^\infty(\mathbb{R})$ with $q(0) = 0$, $q'(0) = 0$ and $q''(0) \neq 0$. Thus we can write $p(\xi) = \sum_{m\in\mathbb{Z}^2} p_m \exp(im \cdot \xi)$ and $q(s) = q_2 s^2 + O(s^3)$.

We look for generalised travelling wave solutions in a direction $k \in S^1_A$ with speed $c \in \mathbb{R}$ of the form,

$$u(x,t) = v\left(x \cdot k - ct, \frac{x}{\varepsilon}\right),$$

where the profile function $v = v(\tau, \xi)$ is a function $v : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ which is periodic in ξ with periodic cell $[0, 2\pi]^2$.

We will study the existence of these generalised travelling waves as c and δ vary around zero by looking at the bifurcations which occur. Thus we get the following result:

Theorem 3.6.1. *Let, $|\delta|$, $|c|$ and $\varepsilon > 0$ be sufficiently small, and $p \in H^2(T^2)$ be such that*

$$\sum_{m \in \mathbb{Z}^2 \setminus \{0\}} \frac{p_{-m}}{\mu_m^+ - \mu_m^-} \left(\frac{1}{\mu_m^+ \left(\mu_m^+ + 2\delta\right) + 4c} + \frac{1}{\mu_m^- \left(\mu_m^- + 2\delta\right) + 4c} \right) \neq 0,$$

where

$$\mu_m^{\pm} = \frac{-\left(c + \frac{2i}{\varepsilon} k^T A m\right) \pm \sqrt{\left(c + \frac{2i}{\varepsilon} k^T A m\right)^2 + \frac{4}{\varepsilon^2} m^T A m - 4\delta}}{2}$$

and $\int_{T^2} p(s)\,ds \neq 0$. Then there exists a generic bifurcation diagram in c and δ for the generalised travelling wave solution v (up to reflection in δ) which is shown in figure 3-4.

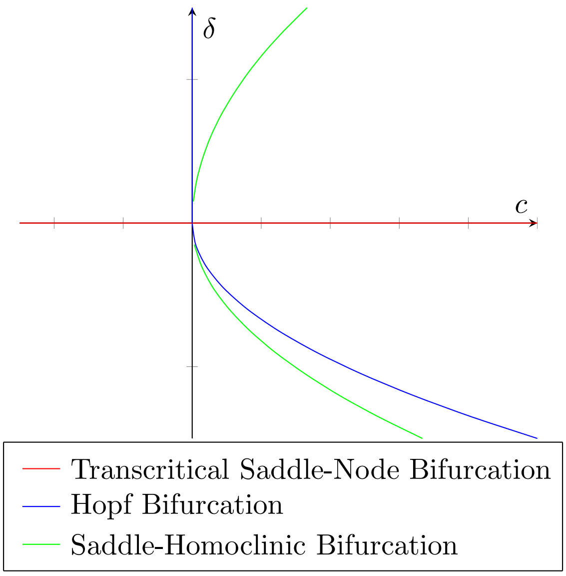


Figure 3-4: *Generic bifurcation diagram for generalised travelling wave solutions.*

Furthermore the bifurcations that occur produce generalised travelling wave solutions of different types.

1. *At the transcritical saddle-node bifurcation two equilibria exchange stability and get generalised travelling wave solution corresponding to a heteroclinic connection between equilibria is created. (These are the solutions found in Theorem 2.1.1.)*

2. *At the Hopf bifurcation we get the creation of a generalised travelling wave solution with a periodic structure in τ i.e.*

$$v(\tau + T, \xi) = v(\tau, \xi),$$

for some $T > 0$.

3. *At the saddle-homoclinic bifurcation we get the creation of a generalised travelling wave solution with a homoclinic structure in τ i.e.*

$$v(\tau, \xi) \to v_0(\xi) \text{ as } \tau \to \pm\infty.$$

This result is the same result as theorem 3.1.1 with more detail included in the statement, thus the proof of this result is the same as the proof of theorem 3.1.1.

Chapter 4

Convergence of Generalised Travelling Waves as $\varepsilon \to 0$

4.1 Introduction

In chapter 2 we considered a reaction diffusion equation in $\mathbb{R}^2$ with a spatially periodic nonlinearity

$$u_t = \operatorname{div}(A\nabla u) + f\left(\frac{x}{\varepsilon}, u\right), \tag{4.1}$$

where $\varepsilon > 0$ is fixed and A is a symmetric positive definite matrix. For this equation we proved the existence of generalised travelling wave solutions with speed $c \neq 0$ in a direction $k \in S^1_A$ of the form

$$u(x,t) = v^\varepsilon\left(x \cdot k - ct, \frac{x}{\varepsilon}\right),$$

where $v^\varepsilon = v^\varepsilon(\tau, \xi)$ is periodic in ξ with periodic cell $[0, 2\pi]^2$, for certain choices of constant $\mu \in \mathbb{R}$ and nonlinearity f.

During this chapter we will consider the case when $\mu = \delta$ is a small constant. In this case the first part of theorem 2.1.1 guarantees the existence of a generalised travelling wave solution which corresponds to a heteroclinic connection between equilibria, for $|\delta| > 0$ sufficiently small and an appropriate nonlinearity.

For this case we will investigate what happens to these generalised travelling wave solutions as $\varepsilon \to 0$. Thus our aim will be to show that the generalised travelling wave solutions which depend periodically on x can be approximated by a homogeneous travelling wave solution which does not depend periodically on x. This is an example of a homogenisation type result: for background information on the theory of homogenisation see the books by, Bensoussan, Lions and Papanicolaou [3], Cioranescu and Donato [14] or Jikov, Kozlov and Oleinik [30].

4.2 Result

If we assume that the nonlinearity is of the form

$$f\left(\frac{x}{\varepsilon}, u\right) = \delta u + p\left(\frac{x}{\varepsilon}\right) q(u),$$

where $\delta \in \mathbb{R}$, $p \in H^2\left(T^2\right)$ and, $q \in C^2(\mathbb{R})$ with $q(0) = 0$, $q'(0) = 0$ and $q''(0) \neq 0$. Then we have the following result.

Theorem 4.2.1. *Let $c \neq 0$ be fixed, $p_0 := \frac{1}{|T^2|}\int_{T^2} p(s)ds \neq 0$ and $\delta \in \mathbb{R}$ be fixed with $|\delta| > 0$ sufficiently small. Then generalised travelling wave profiles satisfy*

$$v^\varepsilon(\tau, \xi) = v^0(\tau) + O(\varepsilon) \ \text{as } \varepsilon \to 0$$

uniformly on $\mathbb{R}$, where the limiting profile v^0 is a heteroclinic connection between equilibria which satisfies the ordinary differential equation

$$v^0_\tau = -\frac{\delta v^0 + p_0 q(v^0)}{c}.$$

The proof of this result will be based on the method we used to construct generalised travelling wave solutions in chapters 2 and 3. Thus the key idea will be to formulate the problem as a spatial dynamical system in such a way that we can control how the reduction map for the local centre manifold changes as $\varepsilon \to 0$. Once this is done we will obtain our desired result by showing that the dynamics on the centre manifold converge as $\varepsilon \to 0$.

Proof of Theorem 4.2.1

The first step towards proving theorem 4.2.1 is to formulate the problem in such a way that we can remove the singular dependence ε while controlling how the spectral gap either side of the imaginary axis changes as $\varepsilon \to 0$. In order to do this we look at rescaled generalised travelling waves solutions

$$u(x, y) = v^\varepsilon\left(x \cdot k - ct, \frac{x}{\varepsilon}\right) = w^\varepsilon\left(\frac{x \cdot k - ct}{\varepsilon}, \frac{x}{\varepsilon}\right),$$

where $w^\varepsilon = w^\varepsilon\left(\tilde{\tau}, \xi\right)$ is periodic in ξ with periodic cell $[0, 2\pi]^2$.

The idea will be to prove the existence of these rescaled generalised travelling wave solutions and determine what happens to them as $\varepsilon \to 0$. This will then tell us what happens to the generalised travelling wave solutions as $\varepsilon \to 0$.

Thus if we substitute this ansatz into the reaction diffusion equation (4.1) we get an equation in terms of the profile function w^ε

$$0 = \frac{1}{\varepsilon^2}\left(\mathrm{div}_\xi \left(A\nabla_\xi w^\varepsilon\right) + 2k^T A\nabla_\xi w^\varepsilon_{\tilde{\tau}} + w^\varepsilon_{\tilde{\tau}\tilde{\tau}}\right) + \frac{c}{\varepsilon} w^\varepsilon_{\tilde{\tau}} + \delta w^\varepsilon + p\left(\xi\right) q(w^\varepsilon),$$

which can be rearranged to get

$$0 = \mathrm{div}_\xi \left(A\nabla_\xi w^\varepsilon\right) + 2k^T A\nabla_\xi w^\varepsilon_{\tilde{\tau}} + w^\varepsilon_{\tilde{\tau}\tilde{\tau}} + c\varepsilon w^\varepsilon_{\tilde{\tau}} + \varepsilon^2 \left(\delta w^\varepsilon + p\left(\xi\right) q(w^\varepsilon)\right). \tag{4.2}$$

Notice that in the second of these equations we have removed all the singular dependence on ε.

Now to prove the existence of rescaled generalised travelling wave solutions we formulate this equation as a spatial dynamical system treating $\tilde{\tau}$ as the time variable and δ as a extra dependent variable. Thus if we let $W^\varepsilon = (w^\varepsilon, w^\varepsilon_{\tilde{\tau}})$ then equation (4.2) gives us the spatial dynamical system

$$\begin{aligned} W^\varepsilon_{\tilde{\tau}} &= \mathcal{J}_\varepsilon W^\varepsilon + \varepsilon^2 F\left(W^\varepsilon, \delta\right) \\ \delta_{\tilde{\tau}} &= 0, \end{aligned} \tag{4.3}$$

where

$$\mathcal{J}_\varepsilon = \begin{pmatrix} 0 & 1 \\ -\mathrm{div}_\xi\left(A\nabla_\xi \cdot\right) & -c\varepsilon - 2k^T A \nabla_\xi \end{pmatrix}$$

and

$$F\left(W^\varepsilon, \delta\right) = \begin{pmatrix} 0 \\ -\delta w^\varepsilon - pq(w^\varepsilon) \end{pmatrix}.$$

Notice that this spatial dynamical system (4.3) is the same as the spatial dynamical system (2.6) in chapter 2 if we choose the wave speed c to be $c\varepsilon$ in the travelling wave ansatz (2.3), set ε to equal to 1 in the linear part of (2.6) and multiplying the nonlinearity by ε^2.

Thus via a similar calculation to the one done in chapter 2 we see that the linear operator $\mathcal{J}_\varepsilon$ has eigenvalues

$$\hat{\lambda}^\pm_{m,\varepsilon} = \frac{-\left(c\varepsilon + 2\mathrm{i}k^T Am\right) \pm \sqrt{\left(c\varepsilon + 2\mathrm{i}k^T Am\right)^2 + 4m^T Am}}{2}, \tag{4.4}$$

with associated eigenfunctions

$$W^\pm_m = \begin{pmatrix} 1 \\ \hat{\lambda}^\pm_{m,\varepsilon} \end{pmatrix} \exp\left(\mathrm{i}m \cdot \xi\right).$$

Therefore if we define the spaces

$$Z_\varepsilon := \left\{ \sum_{m\in\mathbb{Z}^2} \alpha_m^\pm W_m^\pm : \overline{\alpha_m^\pm} = \alpha_{-m}^\pm, \sum_{m\in\mathbb{Z}^2} \left|\alpha_m^\pm\right|^2 \left(1 + \left|\hat{\lambda}_{m,\varepsilon}^\pm\right|^4\right) < \infty \right\}$$

and

$$X_\varepsilon := \left\{ \sum_{m\in\mathbb{Z}^2} \alpha_m^\pm W_m^\pm : \overline{\alpha_m^\pm} = \alpha_{-m}^\pm, \sum_{m\in\mathbb{Z}^2} \left|\alpha_m^\pm\right|^2 \left(1 + \left|\hat{\lambda}_{m,\varepsilon}^\pm\right|^6\right) < \infty \right\},$$

then from proposition 2.3.1 we have a centre manifold reduction for each fixed $\varepsilon > 0$ and thus the following result holds:

Proposition 4.2.2. *For $\varepsilon > 0$ fixed there exist a finite dimensional subspace $X_c^\varepsilon \times \mathbb{R} \subset X_\varepsilon \times \mathbb{R}$ and a projection π_c onto X_c^ε. Letting $X_h^\varepsilon = (Id - \pi_c)(X_\varepsilon)$ there exists a neighbourhood of the origin $\Omega_\varepsilon \subset X_\varepsilon \times \mathbb{R}$ and a map $\psi^\varepsilon \in C_b^2(X_c^\varepsilon \times \mathbb{R}, X_h^\varepsilon)$ with $\psi^\varepsilon(0,0) = 0$ and $D\psi^\varepsilon(0,0) = 0$. Such that if $(W^{c,\varepsilon}, \delta) : I \to X_c \times \mathbb{R}$ solves*

$$\begin{aligned} W_\tau^{c,\varepsilon} &= \mathcal{J}_\varepsilon W^{c,\varepsilon} + \varepsilon^2 \pi_c F\left(W^{c,\varepsilon} + \psi^\varepsilon\left(W^{c,\varepsilon}, \delta\right), \delta\right), \\ \delta_\tau &= 0 \end{aligned}$$

for some interval $I \subset \mathbb{R}$, and $(W^\varepsilon, \delta)(\tau) = (W^{c,\varepsilon}(\tau) + \psi^\varepsilon(W^{c,\varepsilon}(\tau), \delta(\tau)), \delta(\tau)) \in \Omega_\varepsilon$ for all $\tau \in I$ then (W^ε, δ) solves

$$\begin{aligned} W_\tau^\varepsilon &= \mathcal{J}_\varepsilon W^\varepsilon + \varepsilon^2 F(W^\varepsilon, \delta) \\ \delta_\tau &= 0. \end{aligned}$$

Furthermore, from lemmata 2.3.9 and 2.3.13, we have that $\mathcal{J}_\varepsilon \in \mathcal{L}(X_\varepsilon, Z_\varepsilon)$, the nonlinearity $F \in C^2(X_\varepsilon \times \mathbb{R}, X_\varepsilon)$ and $\sigma(\mathcal{J}_\varepsilon) = \left\{\hat{\lambda}_m^\pm : m \in \mathbb{Z}^2\right\}$.

Thus, from the equation for the eigenvalues (4.4), it follows that the only eigenvalues with zero real part are $\hat{\lambda}_{o,\varepsilon}^+ = 0$ if $c > 0$ and $\hat{\lambda}_{0,\varepsilon}^- = 0$ if $c < 0$, so we have that

$$X_c^\varepsilon = \begin{cases} \text{Span}_\mathbb{R}\left\{W_0^+\right\} & \text{if } c > 0 \\ \text{Span}_\mathbb{R}\left\{W_0^-\right\} & \text{if } c < 0 \end{cases} = \text{Span}_\mathbb{R}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\},$$

and by lemma 2.3.16 there exists a projection $\pi_c \in \mathcal{L}(Z_\varepsilon, X_\varepsilon)$ onto X_c^ε. Also if we define the sets and spans of eigenfunctions

$$\begin{aligned} \mathcal{S}^s &= \left\{U_m^\pm | \text{Re}\hat{\lambda}_{m,\varepsilon}^\pm < 0\right\}, \qquad S^s = \text{span}_\mathbb{C}\left\{\mathcal{S}^s\right\} \cap Z_\varepsilon, \\ \mathcal{S}^u &= \left\{U_m^\pm | \text{Re}\hat{\lambda}_{m,\varepsilon}^\pm > 0\right\}, \qquad S^u = \text{span}_\mathbb{C}\left\{\mathcal{S}^u\right\} \cap Z_\varepsilon, \end{aligned}$$

then we can define subspaces X_s^ε, X_u^ε and X_h^ε of X and Z_s^ε, Z_u^ε and Z_h^ε of Z as the closures of S^s, S^u and $S^h := S^s \cup S^u$ in X_ε and Z_ε respectively. Furthermore projections π_s, π_u and $\pi_h \in \mathcal{L}(Z_\varepsilon)$ and $\mathcal{L}(X_\varepsilon)$ on to the relevant spaces can be constructed in a similar way to π_c.

Now proposition 4.2.2 tells us that for each fixed $\varepsilon > 0$ there exists a reduction map ψ^ε and an open neighbourhood of the origin Ω_ε for which the centre manifold property holds. Therefore the next step is to understand what happens to the reduction map and the open neighbourhood of the origin as $\varepsilon \to 0$.

We answer this question by showing that there exists a fixed open neighbourhood of the origin $\Omega \subset X_\varepsilon \times \mathbb{R}$, for which together with ψ^ε the centre manifold property holds, and the reduction map ψ^ε converges to zero uniformly as $\varepsilon \to 0$.

To verify this we will carefully construct the reduction map and show that it has the properties we want.

The first step towards this goal is to understand how the spectral gap either side of the imaginary axis depends on ε. Therefore we need to find a lower bound on the absolute values of the real parts of the non-zero eigenvalues.

Now if $m = 0$ then either $\hat{\lambda}_0^- = -c\varepsilon$ or $\hat{\lambda}_0^+ = -c\varepsilon$ depending on the sign of c and thus the absolute value is bounded below by $|c|\,\varepsilon$. On the other hand if $m \neq 0$ then we need to estimate the absolute value of

$$\mathrm{Re}\hat{\lambda}_{m,\varepsilon}^\pm = \frac{-c\varepsilon \pm \mathrm{Re}\sqrt{(c\varepsilon + 2\mathrm{i}k^T Am)^2 + 4m^T Am}}{2}.$$

We do this by first estimating the absolute value of the real part of the square root. In order to do this we observe that, since $k \in S_A^1$, there exists a vector $k_\perp \in S_A^1$ such that the set $\{k, k_\perp\}$ forms an orthonormal basis for $\mathbb{R}^2$ with respect to the inner product $(x, y)_A = x^T Ay$. So using this basis we can write $m = (m, k)_A k + (m, k_\perp)_A k_\perp$ for all $m \in \mathbb{Z}^2$ and then it follows that

$$m^T Am = (m, k)_A^2 + (m, k_\perp)_A^2.$$

Therefore have that for all $m \neq 0$ and $\varepsilon \in (0,1)$

$$\begin{aligned}
&\left|\mathrm{Re}\sqrt{(c\varepsilon + 2\mathrm{i}k^T Am)^2 + 4m^T Am}\right| \\
&\quad = \left|\mathrm{Re}\sqrt{c^2\varepsilon^2 + 4(m,k_\perp)_A^2 + 4\mathrm{i}c\varepsilon(m,k)_A}\right| \\
&\quad = \frac{1}{\sqrt{2}}\sqrt{\sqrt{\left(c^2\varepsilon^2 + 4(m,k_\perp)_A^2\right)^2 + \left(4c\varepsilon(m,k)_A\right)^2} + c^2\varepsilon^2 + 4(m,k_\perp)_A^2} \\
&\quad \geqslant \frac{1}{\sqrt{2}}\sqrt{\sqrt{c^4\varepsilon^4 + 8c^2\varepsilon^2 m^T Am} + c^2\varepsilon^2} \\
&\quad \geqslant \left(\sqrt{\frac{c^2 + \sqrt{c^4 + 8c^2 m^T Am}}{2}}\right)\varepsilon \qquad \text{(as } 0 < \varepsilon < 1\text{)}.
\end{aligned}$$

Now, as A is a symmetric positive definite matrix, there exists an $m_\star \in \mathbb{Z}^2$ such that $m_\star^T Am_\star = \min_{m \in \mathbb{Z}^2 \setminus \{0\}} m^T Am > 0$, and thus it follows from the above estimate that

$$\left|\mathrm{Re}\sqrt{(c\varepsilon + 2\mathrm{i}k^T Am)^2 + 4m^T Am}\right| \geqslant \left(\sqrt{\frac{c^2 + \sqrt{c^4 + 8c^2 m_\star^T Am_\star}}{2}}\right)\varepsilon,$$

for all $m \in \mathbb{Z}^2 \setminus \{0\}$. From this estimate and the fact that the only other non-zero eigenvalue is equal to $-c\varepsilon$ we get that

$$\left|\mathrm{Re}\hat{\lambda}_{m,\varepsilon}^{\pm}\right| \geqslant \underbrace{\min\left\{\left(\sqrt{\frac{c^2 + \sqrt{c^4 + 8c^2 m_\star^T Am_\star}}{2}} - |c|\right), |c|\right\}}_{=:\hat{C}}\varepsilon,$$

for all $\mathrm{Re}\hat{\lambda}_{m,\varepsilon}^{\pm} \neq 0$. Thus we see that the size of the spectral gaps either side of the imaginary axis go to zero with order ε as $\varepsilon \to 0$.

Hence we have determined how the spectral gap depends on ε, we can now use this information to determine how the solution operator for the affine problem on X_h^ε, which was described in hypothesis (H3) of the centre manifold theorem A.0.3, depends on ε.

Now if we let $\gamma(\varepsilon) := \min\left\{\mathrm{Re}\hat{\lambda}_{m,\varepsilon}^{\pm} : \mathrm{Re}\hat{\lambda}_{m,\varepsilon}^{\pm} \neq 0\right\}$, then the affine problem is the following; for each $\eta \in [0, \gamma(\varepsilon))$ find a map $K_h^\varepsilon \in \mathcal{L}(C_\eta(\mathbb{R}, X_h^\varepsilon))$ such that if $W^{h,\varepsilon} = K_h^\varepsilon f$, then $W^{h,\varepsilon}$ is the unique solution in $C_\eta(\mathbb{R}, X_h^\varepsilon)$ of

$$W_{\tilde{\tau}}^{h,\varepsilon} = \mathcal{J}_\varepsilon W^{h,\varepsilon} + f.$$

From lemma 2.3.19 we have that there exist exponentially decaying C_0-semigroups $\hat{T}_s(\tilde{\tau})$ and $\hat{T}_u(\tilde{\tau})$ on Z_s^ε and Z_u^ε which decay with rate $\gamma(\varepsilon)$. Therefore from the proof of lemma 2.3.21 we have that K_h^ε exists

$$K_h^\varepsilon f(\tilde{\tau}) = \int_{-\infty}^{\tilde{\tau}} \hat{T}_s(\tilde{\tau} - \sigma)\pi_s f(\sigma)\mathrm{d}\sigma - \int_{\tilde{\tau}}^{\infty} T_u(\sigma - \tilde{\tau})\pi_u f(\sigma)\mathrm{d}\sigma,$$

and we have the estimate

$$\|K_h^\varepsilon\|_{\mathcal{L}(C_\eta(\mathbb{R},X_h^\varepsilon))} \leqslant \frac{2}{\gamma(\varepsilon) - \eta}.$$

With this information we are now in a position to start constructing the reduction map for the local centre manifold. In the following calculations we will give a brief outline of how the reduction map is constructed in order to determine how it depends on ε. More detail of the construction can be found in the proof of [47, Theorem 1].

The first step in constructing the reduction map for the local centre manifold is to apply a smooth cut-off function to the nonlinearity to get the equation

$$\begin{aligned} W_{\tilde{\tau}}^\varepsilon &= \mathcal{J}_\varepsilon W^\varepsilon + \varepsilon^2 \chi\left(W^\varepsilon, \delta\right) F(W^\varepsilon, \delta) =: \mathcal{J}_\varepsilon W^\varepsilon + \varepsilon^2 G(W^\varepsilon, \delta) \\ \delta_{\tilde{\tau}} &= 0, \end{aligned} \tag{4.5}$$

where χ is a smooth cut-off function which is independent of ε, such that G is a C^1-bounded and globally Lipschitz function, and $\chi(W^\varepsilon, \delta) = 1$ for $(W^\varepsilon, \delta) \in \Omega \subset X_\varepsilon \times \mathbb{R}$ an open neighbourhood of the origin (details of how to construct this cut-off function can be found in the proof of [47, Theorem 3]).

Now that we have equation (4.5) the main idea is to formulate this equation as an abstract equation on a suitable Banach space. We are then able to use Banach's contraction mapping principle to find solutions to this abstract problem which are then used to construct the reduction map.

Thus if we let $\zeta = \gamma(\varepsilon)/2$ then we can formulate (4.5) as the following abstract problem; for $\delta \in \mathbb{R}$ fixed find $W^\varepsilon \in C_\zeta(\mathbb{R}, X_\varepsilon)$ such that

$$W^\varepsilon = \pi_c W^\varepsilon(0) + \varepsilon^2 K^\varepsilon G(W^\varepsilon, \delta),$$

where $K^\varepsilon \in \mathcal{L}\left(C_\zeta(\mathbb{R}, X_\varepsilon)\right)$ is defined by

$$K^\varepsilon V(\tilde{\tau}) = \int_0^{\tilde{\tau}} \pi_c V(s)\mathrm{d}s + K_h^\varepsilon\left(\pi_h V\right)(\tilde{\tau})$$

for which we have the estimate,

$$\|K^\varepsilon\|_{\mathcal{L}(C_\zeta(\mathbb{R},X_\varepsilon))} \leqslant \frac{1}{\zeta} + \frac{2}{\gamma(\varepsilon) - \zeta} \leqslant \frac{6}{\gamma(\varepsilon)} \leqslant \frac{6}{\hat{C}\varepsilon}.$$

Now if we consider the equation

$$W^{\varepsilon} = W^0 + \varepsilon^2 K^{\varepsilon} G(W^{\varepsilon}, \delta),$$

for $W^0 \in C_{\zeta}(\mathbb{R}, X_{\varepsilon})$, then for $\varepsilon > 0$ sufficiently small we have that

$$\left|\varepsilon^2 K^{\varepsilon} G\right|_{\text{lip}} \leqslant \frac{6\varepsilon}{\hat{C}} |G|_{\text{lip}} < 1,$$

and thus it follows from Banach's contraction mapping theorem that there exists a map $\Psi^{\varepsilon} : C_{\zeta}(\mathbb{R}, X_{\varepsilon}) \times \mathbb{R} \to C_{\zeta}(\mathbb{R}, X_{\varepsilon})$ such that,

$$\Psi^{\varepsilon}(W, \delta) = W^0 + \varepsilon^2 K^{\varepsilon} G(\Psi^{\varepsilon}(W, \delta), \delta). \tag{4.6}$$

Then using this map Ψ^{ε} we have constructed, we define the reduction map to be

$$\psi^{\varepsilon}(W^c, \delta) := \pi_h \Psi^{\varepsilon}(W^c, \delta)(0) = \varepsilon^2 K_h^{\varepsilon}(\pi_h G(\Psi^{\varepsilon}(W^c, \delta), \delta))(0). \tag{4.7}$$

For which we have the estimate

$$\begin{aligned} \|\psi^{\varepsilon}(W^c, \delta)\|_X &\leqslant \varepsilon^2 \|K^{\varepsilon}\|_{\mathcal{L}(C_{\zeta}(\mathbb{R}, X_{\varepsilon}))} \|\pi_h\|_{\mathcal{L}(X_{\varepsilon})} \|G\|_{C(X_{\varepsilon}, X_{\varepsilon})} \\ &\leqslant \frac{6\varepsilon}{\hat{C}} \|G\|_{C(X_{\varepsilon}, X_{\varepsilon})} \to 0 \text{ as } \varepsilon \to 0. \end{aligned} \tag{4.8}$$

Now it is proved in the proof of [47, Theorem 1] that the reduction map constructed in this way is in $C_b^{0,1}(X_c^{\varepsilon} \times \mathbb{R}, X_h^{\varepsilon})$. Then, since $G \in C^2(X_{\varepsilon} \times \mathbb{R}, X_{\varepsilon})$ and the Lipschitz constant of $\varepsilon^2 G$ converges to zero faster than the spectral gaps either side of the imaginary axis as $\varepsilon \to 0$, it follows from [47, Theorem 2] that $\psi^{\varepsilon} \in C_b^2(X_c^{\varepsilon} \times \mathbb{R}, X_h^{\varepsilon})$ for sufficiently small $\varepsilon > 0$.

We are also able to determine how the derivative with respect to W^c of the reduction map ψ^{ε} behaves as $\varepsilon \to 0$. To do this we first need to find the derivative of Ψ^{ε} with respect to W. This is found by differentiating equation (4.6) with respect to W to get

$$D_W \Psi^{\varepsilon}(W, \delta) = Id + \varepsilon^2 K^{\varepsilon} DG(\Psi^{\varepsilon}(W, \delta)) D_W \Psi^{\varepsilon}(W, \delta),$$

which, for sufficiently small $\varepsilon > 0$, we can rearrange to obtain

$$\begin{aligned} D_W \Psi^{\varepsilon}(W, \delta) &= \left(Id - \varepsilon^2 K^{\varepsilon} DG(\Psi^{\varepsilon}(W, \delta))\right)^{-1} \\ &= \sum_{n=0}^{\infty} \varepsilon^{2n} (K^{\varepsilon} DG(\Psi^{\varepsilon}(W, \delta)))^n. \end{aligned}$$

Therefore if we differentiate equation (4.7) with respect to W^c we see that,

$$D_{W^c}\psi^\varepsilon\left(W^c,\delta\right) = \varepsilon^2 K_h^\varepsilon \pi_h D_W G\left(\Psi^\varepsilon\left(W^c,\delta\right),\delta\right) D\Psi^\varepsilon\left(W^c,\delta\right),$$

which will tend to 0 uniformly as $\varepsilon \to 0$ by a similar estimate to (4.8).

Thus we have shown that for all sufficiently small $\varepsilon > 0$ there exist a local centre manifold reduction with a fixed open neighbourhood of the origin $\Omega \subset X_\varepsilon \times \mathbb{R}$ and a reduction map $\psi^\varepsilon(W^c,\delta) = O(\varepsilon)$ as $\varepsilon \to 0$.

Now to understand what happen to the rescaled travelling wave solutions as $\varepsilon \to 0$ it just remains to investigate the dynamics on the centre manifold. This will allow us to show that there exists a rescaled travelling wave solution which corresponds to a heteroclinic connection between equilibria for all $\varepsilon > 0$ sufficiently small and determine what happens to these connections as $\varepsilon \to 0$.

Thus we want to find solutions $(W^{c,\varepsilon},\delta)$ of the ordinary differential equation on $X_c^\varepsilon \times \mathbb{R}$

$$\begin{aligned} W^{c,\varepsilon}_{\tilde{\tau}} &= \varepsilon^2 \pi_c F\left(W^{c,\varepsilon} + \psi^\varepsilon\left(W^{c,\varepsilon},\delta\right),\delta\right) \\ \delta_{\tilde{\tau}} &= 0, \end{aligned} \tag{4.9}$$

such that $\left(W^{c,\varepsilon}(\tilde{\tau}) + \psi^\varepsilon\left(W^{c,\varepsilon}(\tilde{\tau}),\delta\right),\delta\right) \in \Omega$ for all $\tilde{\tau} \in \mathbb{R}$, which correspond to heteroclinic connections between equilibria and determine what happens to these solutions as $\varepsilon \to 0$. Now as the second of these equations just tells us that $\delta \in \mathbb{R}$ is a fixed constant, we can treat δ as a fixed constant and just deal with the first equation.

As

$$X_c^\varepsilon = \begin{cases} \operatorname{Span}_{\mathbb{R}}\left\{W_0^+\right\} & \text{if } c > 0 \\ \operatorname{Span}_{\mathbb{R}}\left\{W_0^-\right\} & \text{if } c < 0 \end{cases},$$

we will assume without loss of generality that $c > 0$ and $X_c = \operatorname{Span}_{\mathbb{R}}\left\{W_0^+\right\}$ throughout this section, if $c < 0$ then the calculation from this section will still give the result with the roles of W_0^+ and W_0^- interchanged.

Thus if we let $W^c = yW_0^+$ and

$$\psi^\varepsilon\left(yW_0^+,\delta\right) =: \begin{pmatrix} \phi_1^\varepsilon(y,\delta) \\ \phi_2^\varepsilon(y,\delta) \end{pmatrix},$$

then the first equation of equation (4.9) becomes the ordinary differential equation

$$\begin{aligned} y_{\tilde{\tau}} W_0^+ &= \varepsilon^2 \pi_c \begin{pmatrix} 0 \\ -\delta\left(y + \phi_1^\varepsilon(y,\delta)\right) - pq\left(y + \phi_1^\varepsilon(y,\delta)\right) \end{pmatrix} \\ &= \varepsilon\left(\frac{-\delta y - p_0 q(y)}{c}\right) W_0^+ + \varepsilon^2 \pi_c \begin{pmatrix} 0 \\ -\delta\phi_1^\varepsilon(y,\delta) - p\left(q\left(y + \phi_1^\varepsilon(y,\delta)\right) - q(y)\right) \end{pmatrix}. \end{aligned}$$

and so if we let $\theta = \varepsilon\tilde{\tau}$ then this equation becomes

$$y_\theta W_0^+ = \left(\frac{-\delta y - p_0 q(y)}{c}\right) W_0^+ + \varepsilon\pi_c \begin{pmatrix} 0 \\ -\delta\phi_1^\varepsilon(y,\delta) - p\left(q\left(y + \phi_1^\varepsilon(y,\delta)\right) - q(y)\right) \end{pmatrix}. \tag{4.10}$$

Now the first thing we want to show is that for $\delta \in \mathbb{R}$ with $|\delta| > 0$ sufficiently small fixed and $\varepsilon > 0$ sufficiently small this equation has a heteroclinic connection between two equilibria. We will do this by using a perturbation argument, thus we start by considering the equation

$$y_\theta = -\left(\frac{\delta y + p_0 q(y)}{c}\right). \tag{4.11}$$

If we use the rescaling

$$\tilde{y} = \frac{y}{\delta} \text{ and } \tilde{\theta} = \delta\theta,$$

then this equation becomes

$$\tilde{y}_{\tilde{\theta}} = -\frac{\tilde{y} + q_2 p_0 \tilde{y}^2 + \delta O\left(\tilde{y}^3\right)}{c}, \tag{4.12}$$

which for $|\delta|$ small can be viewed as a perturbation of the equation

$$\tilde{y}_{\tilde{\theta}} = -\frac{\tilde{y} + q_2 p_0 \tilde{y}^2}{c}. \tag{4.13}$$

Now this equation has a heteroclinic orbit between the stable hyperbolic equilibrium $\tilde{y} = 0$ and unstable hyperbolic equilibrium $\tilde{y} = -1/q_2 p_0$ given by

$$\tilde{y}_1(\tilde{\theta}) = \begin{cases} \frac{-e^{-\frac{\tilde{\theta}}{c}}}{1 + q_2 p_0 \mathrm{e}^{-\frac{\tilde{\theta}}{c}}} & \text{if } p_0 q_2 > 0, \\ \frac{e^{-\frac{\tilde{\theta}}{c}}}{1 - q_2 p_0 \mathrm{e}^{-\frac{\tilde{\theta}}{c}}} & \text{if } p_0 q_2 < 0. \end{cases}$$

Thus, since heteroclinic connections between stable and unstable hyperbolic equilibria are structurally stable, it follows that for $|\delta|$ sufficiently small there will be a heteroclinic connection between two equilibria for the perturbed system (4.12). If we denote this heteroclinic by $\tilde{y}_2(\tilde{\theta})$ then

$$\tilde{y}_2(\tilde{\theta}) \to \tilde{y}_2^{\pm} \text{ as } \tilde{\theta} \to \pm\infty,$$

and if we undo the rescaling we performed we get a heteroclinic connection for equation (4.11) given by, $y^0(\theta) = \delta\tilde{y}_2(\delta\theta)$.

This heteroclinic connection is a structurally stable solution for the ordinary differential equation

$$y_\theta^0 W_0^+ = \left(\frac{-\delta y^0 - p_0 q(y^0)}{c}\right) W_0^+, \tag{4.14}$$

which is part of equation (4.10). Therefore to prove the existence of a heteroclinic connection for equation (4.10) all we need to show is that the remaining parts of equation (4.10) are a small C^1-perturbation of equation (4.14) for $\varepsilon > 0$ sufficiently small, since if this is the case then the heteroclinic will persist for small values of $\varepsilon > 0$.

Thus we need to show that

$$P^\varepsilon(y) := \varepsilon\pi_c \begin{pmatrix} 0 \\ -\delta\phi_1^\varepsilon(y,\delta) - p\left(q\left(y + \phi_1^\varepsilon\left(y,\delta\right)\right) - q(y)\right) \end{pmatrix}$$

and its derivative tend to zero as $\varepsilon \to 0$ in a neighbourhood of our heteroclinic connection y^0.

If we choose an $\varepsilon_0 > 0$ then there exist intervals I and J such that

$$\left\{y^0(\theta) : \theta \in \mathbb{R}\right\} \subset\subset I$$

and

$$I \cup \{y + \phi_1^\varepsilon(y,\delta) : y \in I, \varepsilon \in (0,\varepsilon_0)\} \subset\subset J.$$

Then since $q \in C^2(\mathbb{R})$ it will be a Lipschitz function on the interval J with Lipschitz constant L_q. So for $y \in I$ we have that

$$|q(y + \phi_1^\varepsilon(y,\delta)) - q(y)| \leqslant L_q\,|\phi_1^\varepsilon(y,\delta)| = O(\varepsilon) \text{ as } \varepsilon \to 0,$$

from estimate (4.8). Thus it follows from this estimate together with estimate (4.8) that

$$P^\varepsilon(y) = \varepsilon\pi_c \begin{pmatrix} 0 \\ -\delta\phi_1^\varepsilon(y,\delta) - p\left(q\left(y + \phi_1^\varepsilon\left(y,\delta\right)\right) - q(y)\right) \end{pmatrix} = O(\varepsilon) \text{ as } \varepsilon \to 0,$$

uniformly on I.

Now if we differentiate P^ε with respect to y we get

$$\begin{aligned} D_yP^\varepsilon(y) =& \varepsilon\pi_c \begin{pmatrix} 0 \\ -\delta D_y\phi_1^\varepsilon\left(y,\delta\right) \end{pmatrix} \\ &+ \varepsilon\pi_c \begin{pmatrix} 0 \\ -p\left(q'\left(y + \phi_1^\varepsilon(y,\delta)\right) - q'(y) + q'\left(y + \phi_1^\varepsilon(y,\delta)\right) D_y\phi_1^\varepsilon(y,\delta)\right) \end{pmatrix} \end{aligned}$$

and thus since q is an entire function we have that q' is a Lipschitz function on J, so as $D_y\phi_1^\varepsilon\left(y,\delta\right) \to 0$ uniformly as $\varepsilon \to 0$, it follows by a similar argument to the one used for P^ε, that $D_yP^\varepsilon(y) \to 0$ as $\varepsilon \to 0$ on I.

Thus, since the heteroclinic $y^0(\theta)$ is a structurally stable solution for equation (4.14) and P^ε is a small C^1-perturbation for sufficiently small $\varepsilon > 0$, there will exist a heteroclinic connection between equilibria $y^\varepsilon(\theta)$ which solves equation (4.10). Furthermore it follows from this perturbation argument that the equilibria which the heteroclinic $y^\varepsilon(\theta)$ connects will converge to those that the heteroclinic $y^0(\theta)$ connects as $\varepsilon \to 0$.

Next we want to look at the convergence of these heteroclinic connections y^ε as $\varepsilon \to 0$. For convenience we will define,

$$g(y) := -\frac{\delta y + p_0 q(y)}{c},$$

and

$$h^\varepsilon(y) W_0^+ := P^\varepsilon(y).$$

Then using this notation equation (4.10) becomes

$$y_\theta W_0^+ = g(y) W_0^+ + h^\varepsilon(y) W_0^+,$$

and so the ordinary differential equation we are interested in is

$$y_\theta = g(y) + h^\varepsilon(y).$$

From the perturbation argument we just performed we know that for $\delta \in \mathbb{R}$ with $|\delta|$ sufficiently small and $\varepsilon > 0$ sufficiently small there exists a heteroclinic connection between two equilibria $y^\varepsilon(\theta)$ such that

$$y_\theta^\varepsilon = g\left(y^\varepsilon\right) + h^\varepsilon\left(y^\varepsilon\right) \tag{4.15}$$

and

$$y^\varepsilon(\theta) \to y_\pm^\varepsilon \text{ as } \theta \to \pm\infty.$$

Also from our study of equation (4.11) in the perturbation argument we know that there exists a heteroclinic connection between equilibria $y^0(\theta)$ such that

$$y_\theta^0 = g\left(y^0\right) \tag{4.16}$$

and

$$y^0(\theta) \to y_\pm^0 \text{ as } \theta \to \pm\infty.$$

Finally the perturbation argument also tells us that the equilibria $y_\pm^\varepsilon \to y_\pm^0$ as $\varepsilon \to 0$.

Now with these facts we will prove that for δ fixed the following convergence result

$$y^\varepsilon = y^0 + O(\varepsilon) \text{ as } \varepsilon \to 0,$$

uniformly on $\mathbb{R}$.

The proof of this result is based on the proof of [43, Theorem 5.3.1] with some minor adaptations to deal with the fact our heteroclinic solutions are defined on the whole of $\mathbb{R}$. The basic idea is to split $\mathbb{R}$ into a series of subintervals and then use a combination of the exponential attraction of solutions near hyperbolic equilibria and Gronwall type arguments to get the convergence we want on each subinterval.

Therefore we start by looking at the behaviour of solutions to (4.16) near the equilibria y^0_+ and y^0_-. Since y^0_+ and y^0_- are hyperbolically stable and unstable equilibria respectively, by [43, Lemma 5.2.7] there exist constants $\rho > 0$, $M > 0$ and $\mu > 0$ such that if $\hat{y}^0$ and $\tilde{y}^0$ solve equation (4.16) then we have the following two properties:

1. If $\hat{y}^0(\theta_0)$ and $\tilde{y}^0(\theta_0) \in B_\rho\left(y^0_+, \mathbb{R}\right)$ for $\theta_0 \in \mathbb{R}$ then

$$\left|\hat{y}^0(\theta) - \tilde{y}^0(\theta)\right| \leqslant Me^{-\mu|\theta-\theta_0|}\left|\hat{y}^0(\theta_0) - \tilde{y}^0(\theta_0)\right| \text{ for all } \theta \geqslant \theta_0. \tag{4.17}$$

2. If $\hat{y}^0(\theta_0)$ and $\tilde{y}^0(\theta_0) \in B_\rho\left(y^0_-, \mathbb{R}\right)$ for $\theta_0 \in \mathbb{R}$ then

$$\left|\hat{y}^0(\theta) - \tilde{y}^0(\theta)\right| \leqslant Me^{-\mu|\theta-\theta_0|}\left|\hat{y}^0(\theta_0) - \tilde{y}^0(\theta_0)\right| \text{ for all } \theta \leqslant \theta_0. \tag{4.18}$$

Also, since $Me^{-\mu s} \to 0$ as $s \to \infty$, there will exist a $T_1 > 0$ such that

$$M \exp\left(-\mu s\right) \leqslant 1/2 \text{ for all } s \geqslant T_1. \tag{4.19}$$

Next we need estimates on when the heteroclinic connection y^0 and the equilibria $y^\varepsilon_\pm$ are close enough to $y^0_\pm$ that the exponential estimates (4.17) and (4.18) hold. Thus, as $y^0(\theta) \to y^0_\pm$ when $\theta \to \pm\infty$, there exists a $T_2 > 0$ such that

$$\left|y^0(\pm\theta) - y^0_\pm\right| < \frac{\rho}{2} \text{ for all } \theta \geqslant T_2, \tag{4.20}$$

and, since $y^\varepsilon_\pm \to y^0_\pm$ as $\varepsilon \to 0$ there exists a $\varepsilon_1 > 0$ such that

$$\left|y^\varepsilon_\pm - y^0_\pm\right| < \rho \text{ for all } \varepsilon \in (0, \varepsilon_1). \tag{4.21}$$

Now if we let $T = \max\{T_1, T_2\}$ then we split $\mathbb{R}$ into a union of finite intervals

$$\mathbb{R} = \bigcup_{n\in\mathbb{Z}} I_n,$$

where

$$I_n = \begin{cases} [nT, (n+1)T] & \text{for } n > 0 \\ [-T, T] & \text{for } n = 0 \\ [(n-1)T, nT] & \text{for } n < 0 \end{cases}.$$

With this splitting of $\mathbb{R}$ into subintervals the first step towards proving the convergence we want is to use a Gronwall type argument to show that we have our desired convergence on the finite interval [-2T,2T]. This argument is based on the proof of [43, Lemma 1.5.3].

In order to show this result we first need to collect some properties of the functions g and h^ε. Due to the fact g is an entire function it follows that if we choose an interval $\hat{I}$ such that

$$\left\{y^0(\theta) : \theta \in \mathbb{R}\right\} \cup \left\{y^\varepsilon(\theta) : \theta \in \mathbb{R}, \varepsilon \in (0, \varepsilon_1)\right\} \subset\subset \hat{I},$$

then g will be a Lipschitz function on $\hat{I}$ with Lipschitz constant L_g. On the other hand using a similar argument to the one we used to show that $P^\varepsilon(y) = O(\varepsilon)$ on I as $\varepsilon \to 0$ we can show that $h^\varepsilon(y) = O(\varepsilon)$ on $\hat{I}$ as $\varepsilon \to 0$ and thus there will exist a constant $M_h > 0$ such that

$$|h^\varepsilon(y)| \leqslant M_h \varepsilon \text{ for all } y \in \hat{I}.$$

Also, since any translation of the heteroclinic connections y^0 and y^ε are also heteroclinic connection, if we choose

$$\tilde{y} \in \left\{y^0(\theta) : \theta \in [-2T, 2T]\right\} \cap \bigcap_{\varepsilon \in (0,\varepsilon_0)} \left\{y^\varepsilon(\theta) : \theta \in [-2T, T]\right\}$$

then by translating the heteroclinic connections y^0 and y^ε we can ensure that $y^0(0) = \tilde{y}$ and $y^\varepsilon(0) = \tilde{y}$.

Therefore for $\theta \in [-2T, 2T]$ and $\varepsilon \in (0, \varepsilon_0)$ we can use equations (4.15) and (4.16), and the fundamental theorem of calculus to write $y^\varepsilon(\theta)$ and $y^0(\theta)$ as

$$y^\varepsilon(\theta) = \tilde{y} + \int_0^\theta g\left(y^\varepsilon(\sigma)\right) + h^\varepsilon\left(y^\varepsilon(\sigma)\right) \mathrm{d}\sigma$$

and

$$y^0(\theta) = \tilde{y} + \int_0^\theta g\left(y^\varepsilon(\sigma)\right) \mathrm{d}\sigma.$$

Then from these representations it follows that

$$\begin{aligned}
\left|y^\varepsilon(\theta) - y^0(\theta)\right| &\leqslant \left|\int_0^\theta \left|g\left(y^\varepsilon(\sigma)\right) - g\left(y^0(\sigma)\right)\right| \mathrm{d}\sigma\right| + \left|\int_0^\theta \left|h^\varepsilon\left(y^\varepsilon(\sigma)\right)\right| \mathrm{d}\sigma\right| \\
&\leqslant L_g \left|\int_0^\theta \left|y^\varepsilon(\sigma) - y^0(\sigma)\right| \mathrm{d}\sigma\right| + |\theta| \, M_h \varepsilon \\
&\leqslant |\theta| \, M_h \varepsilon \exp\left(L_g |\theta|\right) \qquad \text{(by Gronwall's inequality)} \\
&\leqslant 2T M_h \varepsilon \exp\left(2 L_g T\right) =: \Gamma(\varepsilon). \qquad (4.22)
\end{aligned}$$

Thus we have our desired convergence on the interval $[-2T, 2T]$. We know need to deal with the intervals further out.

In order to do this we first need to find out when the heteroclinic y^ε is close enough to $y^0_\pm$ for a solution to (4.16) with initial data on y^ε to have the exponential estimates (4.17) and (4.18) to hold. Hence, since $\Gamma(\varepsilon) \to 0$ as $\varepsilon \to 0$, there exist a $\varepsilon_2 \in (0, \varepsilon_1]$ such that

$$\Gamma(\varepsilon) \leqslant \frac{\rho}{2} \text{ for all } \varepsilon \in (0, \varepsilon_2).$$

Therefore if $\varepsilon \in (0, \varepsilon_2)$ then it follows from the estimates (4.20), (4.21) and (4.22) that

$$\left|y^\varepsilon(\theta) - y^0_+\right| < \rho \text{ for all } \theta \geqslant T$$

and

$$\left|y^\varepsilon(\theta) - y^0_-\right| < \rho \text{ for all } \theta \leqslant -T.$$

With this information we can now show the convergence we want on the rest of $\mathbb{R}$. Thus we start by considering the interval I_n for $n \geqslant 2$. In order to get the estimates we want on this interval we need an intermediate solution, thus we define a solution $y^{0,n}$ of the ordinary differential equation (4.16) which has initial value $y^{0,n}((n-1)T) = y^\varepsilon((n-1)T)$. Then using a Gronwall type argument similar to the one done above we get that

$$\sup_{\theta \in I_n} \left|y^\varepsilon(\theta) - y^{0,n}(\theta)\right| \leqslant \Gamma(\varepsilon).$$

Thus it follows that

$$\begin{aligned}\sup_{\theta \in I_n} \left|y^\varepsilon(\theta) - y^0(\theta)\right| &\leqslant \sup_{\theta \in I_n} \left|y^\varepsilon(\theta) - y^{0,n}(\theta)\right| + \sup_{\theta \in I_n} \left|y^{0,n}(\theta) - y^0(\theta)\right| \\ &\leqslant \Gamma(\varepsilon) + \sup_{\theta \in I_n} \left|y^{0,n}(\theta) - y^0(\theta)\right|,\end{aligned}$$

and, since $\theta \geqslant nT$ for $\theta \in I_n$, $y^0((n-1)T) \in B_\rho\left(y^0_+, \mathbb{R}\right)$ and $y^{0,n}((n-1)T) = y^\varepsilon((n-1)T) \in B_\rho\left(y^0_+, \mathbb{R}\right)$, it follows from the exponential estimate (4.17) that

$$\begin{aligned}\sup_{\theta \in I_n} \left|y^\varepsilon(\theta) - y^0(\theta)\right| &\leqslant \Gamma(\varepsilon) + \sup_{\theta \in I_n} M e^{-\mu|\theta-(n-1)T|} \left|y^\varepsilon((n-1)T) - y^0((n-1)T)\right| \\ &\leqslant \Gamma(\varepsilon) + \frac{1}{2} \sup_{\theta \in I_{n-1}} \left|y^\varepsilon(\theta) - y^0(\theta)\right|,\end{aligned}$$

where the second inequality follows from inequality (4.19), as for $\theta \in I_n$ we have that

$\theta - (n-1)T \geqslant T \geqslant T_2$. Now if we apply this inequality recursively we get that

$$\begin{aligned}\sup_{\theta \in I_n} \left|y^{\varepsilon}(\theta) - y^0(\theta)\right| &\leqslant \left(1 + \frac{1}{2}\right)\Gamma(\varepsilon) + \left(\frac{1}{2}\right)^2 I_{(n-2)} \left|y^{\varepsilon}(\theta) - y^0(\theta)\right| \\ &\leqslant \left(1 + \frac{1}{2} + \cdots + \left(\frac{1}{2}\right)^n\right)\Gamma(\varepsilon) \\ &\leqslant 2\Gamma(\varepsilon).\end{aligned}$$

Notice that the last of these estimates is independent of ε and so will hold for all $n \geqslant 2$. Thus we have the convergence we want on the interval $[-2T, \infty)$.

A similar calculation to the one above will work for the interval $(-\infty, -2T]$. Hence we have that for sufficiently small $\varepsilon > 0$

$$\left|y^{\varepsilon}(\theta) - y^0(\theta)\right| \leqslant 2\Gamma(\varepsilon) \text{ for all } \theta \in \mathbb{R},$$

and

$$y^{\varepsilon} = y^0 + O(\varepsilon) \text{ as } \varepsilon \to 0.$$

From this result we have that equation (4.10) has a heteroclinic connection between equilibria,

$$\hat{W}^{c,\varepsilon}(\theta) = y^{\varepsilon}(\theta)W_0^+ = y^0(\theta)W_0^+ + O(\varepsilon), \text{ as } \varepsilon \to 0.$$

Therefore, as $\theta = \varepsilon\tilde{\tau}$, it follows that the ordinary differential equation on the centre manifold (4.9) has a heteroclinic connection

$$W^{c,\varepsilon}(\tilde{\tau}) = y^{\varepsilon}(\varepsilon\tilde{\tau})W_0^+.$$

Now we need to check that if $\varepsilon > 0$ and $|\delta|$ are sufficiently small then when this solution is mapped onto the centre manifold it stays within Ω and is thus a solution of the spatial dynamical system (4.3). We know from the perturbation argument earlier in this proof that $y^0(\theta) = \delta\tilde{y}_2(\delta\theta)$ and thus it will converge uniformly to 0 as $\delta \to 0$. Therefore, since $y^{\varepsilon} = y^0 + O(\varepsilon)$ as $\varepsilon \to 0$ and, ψ^{ε} is continuous with $\psi^{\varepsilon}(0,0) = 0$, it follows that if $|\delta|$ and $\varepsilon > 0$ are sufficiently small then

$$(W^{\varepsilon}(\tilde{\tau}), \delta) = (W^{c,\varepsilon}(\tilde{\tau}) + \psi^{\varepsilon}\left(W^{c,\varepsilon}(\tilde{\tau}), \delta\right), \delta) \in \Omega \text{ for all } \tilde{\tau} \in \mathbb{R}.$$

Thus $(W^{\varepsilon}, \delta)$ solves the spatial dynamical system (4.3) for all $|\delta|$ and $\varepsilon > 0$ sufficiently small. Hence, since $W^{\varepsilon} = (w^{\varepsilon}, w^{\varepsilon}_{\tau})$, we get the rescaled travelling wave profile

$$w^{\varepsilon}\left(\tilde{\tau}, \xi\right) = y^{\varepsilon}(\varepsilon\tilde{\tau}) + \phi_1^{\varepsilon}\left(y^{\varepsilon}(\varepsilon\tilde{\tau}), \delta\right)(\xi),$$

and from the convergence results we have proved we understand what happens to these rescaled travelling wave solutions as $\varepsilon \to 0$.

Thus it follows that the travelling wave profile is of the form

$$v^{\varepsilon}(\tau, \xi) = w^{\varepsilon}\left(\frac{\tau}{\varepsilon}, \xi\right) = y^{\varepsilon}(\tau) + \phi_1^{\varepsilon}(y^{\varepsilon}(\tau), \delta)(\xi)$$

and therefore, from the convergence of the solutions y^{ε} and of the reduction map ϕ^{ε} it follows that

$$v^{\varepsilon}(\tau, \xi) = y^0(\tau) + O(\varepsilon) =: v^0(\tau) + O(\varepsilon) \text{ as } \varepsilon \to 0,$$

uniformly on $\mathbb{R}$, where v^0 is the limiting travelling wave profile and is a heteroclinic connection between equilibria for the ordinary differential equation,

$$v^0_{\tau} = -\frac{\delta v^0 + p_0 q(v^0)}{c}.$$

Thus we have proved the desired result.

□

Chapter 5

Further Work

In this chapter we will briefly discuss several future directions for the work we have presented in this thesis.

The first natural extension would be to look at systems of reaction diffusion equations instead of the scalar equations we have considered in this thesis. The beauty of this technique is that it does not require a maximum principle, which does not in general exist for systems of reaction diffusion equations. An example of such a problem would be to consider a 2-dimensional reaction diffusion system

$$\begin{aligned} u_t^1 &= \Delta u^1 + \delta_1 u_1 + f_1\left(\frac{x}{\varepsilon}, u^1, u^2\right) \\ u_t^2 &= \Delta u^2 + \delta_2 u_2 + f_2\left(\frac{x}{\varepsilon}, u^1, u^2\right) \end{aligned}$$

and to try and prove the existence of generalised travelling wave solutions

$$\begin{aligned} u^1(x,t) &= v^1\left(x \cdot k - ct, \frac{x}{\varepsilon}\right) \\ u^2(x,t) &= v^2\left(x \cdot k - ct, \frac{x}{\varepsilon}\right). \end{aligned}$$

We could analyse such a problem by using very similar methods to those we introduce in chapter 2 and 3. Thus the idea would be to substitute the generalised travelling wave ansatz into the reaction diffusion system to get a system of equation in terms of the profile functions. Once we have these equations in terms of the profile functions v^1 and v^2 we can then formulate the problem as a spatial dynamical system. Then we can use similar methods to either chapter 2 or 3 to get a centre manifold reduction, depending on which of the parameter δ_1, δ_2 and c we choose to treat as variables. We can then study the dynamics on the finite dimensional centre manifold to find solutions. Notice that depending which of the parameters δ_1, δ_2 and c we choose to treat as variable will effect the dimension of X_c, for example if we treat them all as variable then X_c will be four dimensional. However if we only treat δ_1 as a variable then X_c

will be 1-dimensional and a result similar to the first part of theorem 2.1.1 should hold.

Another extension would be to consider a reaction diffusion equation with a diffusion term which depends periodically on the spatial variable. Specifically we could consider a reaction diffusion equation of the form

$$u_t = \operatorname{div}\left(A\left(\frac{x}{\varepsilon}\right)\nabla u\right) + \mu u + f\left(\frac{x}{\varepsilon}, u\right).$$

The addition of the periodic dependence in the diffusion terms in this equation adds new difficulties to the problem as in general it will no longer be possible to explicitly calculate the eigenvalues and eigenfunction of the linear part of the spatial dynamical system. Thus an interesting question is whether we can extend our method to this case and whether the periodic behaviour in the diffusion term will effect the results we get.

A further question relates to the work we did in chapter 3: In chapter 3 we investigated what happen to the generalised travelling wave solutions as the wave speed c and the parameter δ varied close to zero. Thus a related question would be to investigate what happens to the generalised travelling wave solutions as the wave speed $c \to 0$.

The final question we will mention, is the question of whether we can apply the techniques we have developed in this thesis to other similar equation to prove the existence of generalised travelling wave solution. For example we could try and use the techniques from this thesis to prove the existence of generalised travelling wave solution for the nonlinear Schrödinger's equation

$$\mathrm{i}u_t = \Delta u + \mu u + f\left(\frac{x}{\varepsilon}, u\right),$$

or a reaction diffusion equation with a gradient dependent nonlinearity and periodic dependent linear term

$$u_t = \operatorname{div}(A\nabla u) + \mu\left(\frac{x}{\varepsilon}\right)u + f\left(\frac{x}{\varepsilon}, u, \nabla u\right).$$

The questions I have posed here are just a small selection of the possible directions we could take to extend the work in this thesis.

Appendix A

Centre Manifolds in Infinite Dimensional

In this section we give a brief introduction to the local infinite dimensional centre manifold theorem. The information in this section is taken from the paper by Iooss and Vanderbauwhede [47] and the book by Haragus and Iooss [24].

Let X and Z be Banach spaces such that

$$X \hookrightarrow Z$$

continuously; then we consider equations of the form

$$\dot{x} = \mathcal{A}x + f(x); \tag{A.1}$$

such that the following hypotheses hold:

(H1) $\mathcal{A} \in \mathcal{L}(X, Z)$ and for some $k \geqslant 2$ there exists a neighbourhood of the origin $V \subset X$ such that $f \in C^k(V, X)$ and $f(0) = 0$, $Df(0) = 0$.

(H2) There exists a projection $\pi_c \in \mathcal{L}(Z, X)$ onto a finite dimensional subspace $Z_c = X_c \subset X$ such that $\sigma(\mathcal{A}|_{X_c}) \subset \mathrm{i}\mathbb{R}$ and

$$\mathcal{A}\pi_c x = \pi_c \mathcal{A}x,$$

for all $x \in X$.

Now if we set

$$Z_h := (I - \pi_c)Z \text{ and } X_h := (I - \pi_c)X$$

and, for a Banach space E, we denote the space of exponentially growing Banach space valued continuous functions by

$$C_\eta(\mathbb{R}, E) := \left\{ u \in C(\mathbb{R}, E) : \|u\|_\eta := \sup_{\tau \in \mathbb{R}} \left(\mathrm{e}^{-\eta|\tau|} \|u(\tau)\|_E \right) < \infty \right\}.$$

Then the final hypothesis will be:

(H3) There exists a $\gamma > 0$ such that for each $\eta \in [0, \gamma)$ and $f \in C_\eta(\mathbb{R}, X_h)$ the hyperbolic affine problem

$$\dot{x}_h = \mathcal{A}x_h + f \text{ and } x_h \in C_\eta(\mathbb{R}, X_h)$$

has a unique solution $x_h = K_h f$ where $K_h \in \mathcal{L}(C_\eta(\mathbb{R}, X_h))$ for each $\eta \in [0, \gamma)$ and

$$\|K_h\|_{\mathcal{L}(C_\eta(\mathbb{R}, X_h))} \leqslant \delta(\eta)$$

for some continuous function $\delta : [0, \gamma) \to \mathbb{R}^+$.

If these three hypotheses hold then we can perform a local centre manifold reduction using the following theorem.

Theorem A.0.3. *[47, Theorem 3][24, Theorem 2.9] Let (H1) -(H3) hold then there exists a neighbourhood of the origin $\Omega \subset X$ and a map $\psi \in C_b^k(X_c, X_h)$ with $\psi(0) = 0$ and $D\psi(0) = 0$ such that the following properties hold:*

- *If $x_c : I \to X_c$ is a solution of*

$$\dot{x}_c = \mathcal{A}x_c + \pi_c f(x_c + \psi(x_c)); \tag{A.2}$$

  for some interval I, such that $x(t) := x_c(t) + \psi(x_c(t)) \in \Omega$ for all $t \in I$. Then x is a solution of

$$\dot{x} = \mathcal{A}x + f(x). \tag{A.1}$$

- *If $x : \mathbb{R} \to X$ is a solution of (A.1) such that $x(t) \in \Omega$ for all $t \in \mathbb{R}$ then*

$$(I - \pi_c)x(t) = \psi(x(t))$$

  and $\pi_c x : \mathbb{R} \to X_c$ is a solution of (A.2).

For the proof of this theorem see either [47] or [24].

Remark A.0.4. We call ψ the reduction map and it will satisfy the equation

$$D\psi(x_c)(\mathcal{A}x_c + \pi_c f(x_c + \psi(x_c))) = \mathcal{A}\psi(x_c) + (I - \pi_c)f(x_c + \psi(x_c)).$$

In general we cannot find ψ explicitly, since this would be equivalent to solving (A.1), but we can use this equation to workout terms of the Taylor expansion of ψ.

Appendix B

Conley Index

In this appendix we present a brief introduction to Conley index and some important results related to it. The information in this section is taken from the paper by Conley [15] and the book by Smoller [45].

B.1 Isolating Blocks

In this section we introduce a special kind of set which allows us to define the Conley index. Through out this section let

$$\dot{x} = f(x);$$

for $f : \mathbb{R}^n \to \mathbb{R}^n$ Lipschitz continuous and denote the position at time t of the flow generated by this ordinary differential equation starting at x at time 0 by $x \cdot t$ for $t \in \mathbb{R}$.

Remark B.1.1. While flows generated by ordinary differential equation are the ones we will be interested in, all the definitions and results in the next two sections make since for a general flow $\phi : X \times \mathbb{R} \to X$ on a compact manifold X (for the definition of a general flow see definition B.2.1) with convention that $x \cdot t := \phi(x, t)$.

Definition B.1.2.

- *The closure of a bounded open set $N \subset \mathbb{R}^n$ is called an isolating neighbourhood for f if for each $x \in \partial N$ there exists a $t \in \mathbb{R}$ such that $x \cdot t \notin N$.*
- *A closed invariant set is called an isolated invariant set if it is the maximal invariant set in some isolating neighbourhood.*

We will now define a special kind of isolating neighbourhood called a isolating block but before we do that we need to define the concept of a local section.

Definition B.1.3. *Let $S \subset \mathbb{R}^n$, given $\delta > 0$ we define $h_\delta : S \times (-\delta, \delta) \to \mathbb{R}^n$ by $h_\delta(x,t) = x \cdot t$. Then if there exists some $\delta > 0$ such that h_δ is a homeomorphism then S is called a local section.*

Now we are in a position to define the concept of an isolating block.

Definition B.1.4. *Let B be the closure of a open subset of $\mathbb{R}^n$ then B is an isolating block if there exist local sections S^+ and S^- such that the following properties hold.*

1. $\left(\overline{S^\pm} \setminus S^\pm\right) \cap B = \emptyset$
2. $S^- \cdot (-\delta, \delta) \cap B = (S^- \cap B) \cdot [0, \delta)$
3. $S^+ \cdot (-\delta, \delta) \cap B = (S^+ \cap B) \cdot (-\delta, 0]$
4. *If $x \in \partial B \backslash (S^+ \cup S^-)$ then there exists $\varepsilon_1 < 0$ and $\varepsilon_2 > 0$ such that $x \cdot [\varepsilon_1, \varepsilon_2] \subset \partial B$ and $x \cdot \varepsilon_1 \in S^-$, $x \cdot \varepsilon_2 \in S^+$.*

The concept of an isolating block is illustrated in figure B-1. If B is an isolating block

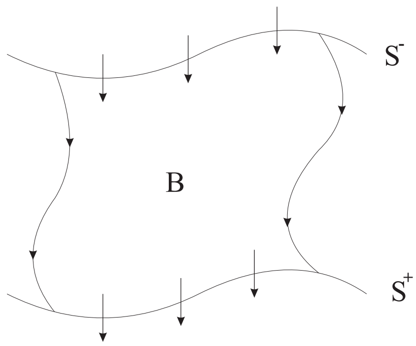


Figure B-1: *Diagram of isolating block.*

then we use the following notation

$$\begin{aligned} b &= \partial B \\ b^+ &= B \cap S^+ \text{ (exit set of } B) \\ b^- &= B \cap S^- \text{ (entry set of } B). \end{aligned}$$

B.1.1 Conley Index

In the previous section we introduced the concept of an isolating block which we will use to define the Conley index. But before we can define the Conley index we need to introduce some topological concepts.

Definition B.1.5. *A topological pair is an ordered pair (X, A) of topological spaces such that A is a closed subspace of X.*

If (X, A) and (Y, B) are topological pairs then a map $f : (X, A) \to (Y, B)$ is a continuous map from X to Y such that $f(A) \subset B$. We will now define what it means for two maps between topological pairs to be homotopy, intuitively what this means is that one map can be continuously deformed into the other.

Definition B.1.6. *Let (X, A) and (Y, B) be topological pairs and suppose $f_0, f_1 : (X, A) \to (Y, B)$, then we say f_0 is homotopy to f_1 and write $f_0 \sim f_1$ if there exists a continuous function $F : (X, A) \times [0, 1] \to (Y, B)$ such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$ for all $x \in X$.*

In actual fact the relation $\sim$ defines an equivalence relation. Next we will use this concept to define what it means for two topological pairs to be homotopy. The idea is that two topological pairs are homotopy if one can be continuously deformed into the other.

Definition B.1.7. *Let (X, A) and (Y, B) be topological pairs then we say that (X, A) is homotopy to (Y, B) and write $(X, A) \sim (Y, B)$ if there exist maps $f : (X, A) \to (Y, B)$ and $g : (Y, B) \to (X, A)$ such that*

$$f \circ g \sim Id_Y \ (\textit{identity on } Y)$$

and

$$g \circ f \sim Id_X \ (\textit{identity on } X).$$

This relation also defines an equivalence relation.

Now if (X, A) is a topological pair then we can define a equivalence relation on X as by $x \sim y$ if $x = y$ or $x, y \in A$. We let $X/\sim$ be the set of all equivalence classes $[x]$ for $x \in X$.

Definition B.1.8. *Let (X, A) be a topological pair then the pointed space X/A is defined to be the topological pair $(X/\sim, [A])$. With the convention that if $A = \emptyset$ then we define this to be the topological pair $(X \sqcup \{x_0\}, x_0)$.*

We now have the necessary topological machinery to define the Conley index.

Definition B.1.9. *Let I be isolated invariant set and let B be an isolating block for I. Then we define the Conley index of I to be*

$$h(I) = \left[B/b^+\right],$$

the homotopy equivalence class of the pointed space B/b^+, where b^+ is the exit set of B.

The Conley index is a well defined index which exists for any isolated invariant set, for details of why this is the case see Conley [15] or Smoller [45]. We will now state a theorem about Conley index which we will make use of in this thesis.

Theorem B.1.10. *Let I_1 and I_2 be isolated invariant sets then $I = I_1 \cup I_2$ is an isolated invariant set and*

$$h(I_1 \sqcup I_2) = h(I_1) \vee h(I_2);$$

where $(X/A) \vee (Y/B) = (X \sqcup Y)/(A \cup B)$.

We will now present several simple example which will illustrate the Conley index and be useful in the calculations we do with Conley index in the main text.

Example B.1.11. Consider the ordinary differential equation

$$\begin{aligned} \dot{x} &= \lambda x \\ \dot{y} &= \lambda y, \end{aligned}$$

for $\lambda < 0$. This equation has a stable equilibrium at 0 and we can take the closed unit ball B as a isolating block for this equilibrium as is shown in figure B-2. Now, since

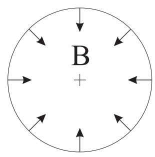


Figure B-2: *Direction of vector field around edge of unit ball for example B.1.11.*

$b^+ = \emptyset$, we get that

$$\begin{aligned} h(\overline{0}) &= [(B \sqcup \{x_0\}, x_0)] \\ &= \left[\left(\{\overline{0}\} \sqcup \{x_0\}, x_0\right)\right], \end{aligned}$$

since the set B can be retracted to the origin.

In actual fact the result from the above example is true in much more: generality any hyperbolic stable equilibruim x will have Conley index $[(\{x\} \sqcup \{x_0\}, x_0)]$.

Example B.1.12. Consider the ordinary differential equation

$$\begin{aligned} \dot{x} &= \lambda x \\ \dot{y} &= \lambda y, \end{aligned}$$

for $\lambda > 0$. Then is equation has a hyperbolic unstable equilibrium at 0 and we can again take the closed unit ball B as the isolating block, as can be seen in figure B-3.

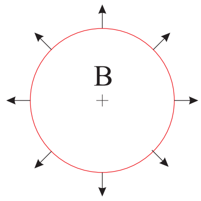
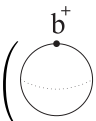


Figure B-3: *Direction of vector field around edge of unit ball for example B.1.12 with exit set b^+ marked in red.*

Now, since $b^+ = \partial B$, we get that

$$h(\underline{0}) = \Bigg[\Bigg($$ 

$$, [b^+]\Bigg)\Bigg] = [(S^2, [b^+])].$$

The last of these is a bit of an abuse of notation since S^2 will have a distinguished point b^+ which is not indicated.

Again the result from the last example is true in general. Now we will finish by looking at a slightly more interesting example that of a hyperbolic saddle equilibrium in 2 dimensions.

Example B.1.13. Consider the ordinary differential equation

$$\begin{aligned} \dot{x} &= \lambda_1 x \\ \dot{y} &= \lambda_2 y, \end{aligned}$$

for $\lambda_1 < 0 < \lambda_2$. Then 0 is a saddle equilibrium and we have to be a little bit cleverer in how we define the isolating block B its construction is shown in figure B-4. Then

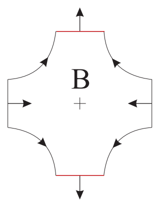
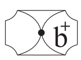
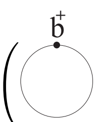


Figure B-4: *Direction of vector field around edge of the unit ball for example B.1.13 with exit set b^+ marked in red.*

we get that the Conley index

$$h(\underline{0}) = \Bigg[\Bigg($$ 

$$, [b^+]\Bigg)\Bigg] = \Bigg[\Bigg($$ 

$$, [b^+]\Bigg)\Bigg] = \big[(S^1, [b^+])\big]\,;$$

since when b^+ becomes a single point we can then retract the resulting set to a circle.

B.2 Continuation

In this section we introduce the important concept of continuation, which allow us to say when two invariant sets for different dynamical systems have same Conley index. To introduce this idea we need to define some concepts.

Throughout this section let Ω be a compact subset of $\mathbb{R}^n$.

Definition B.2.1. *A continuous function $\phi : \Omega \times \mathbb{R} \to \Omega$ is a flow on Ω if*

- $\phi(x, 0) = x$ *for all* $x \in \Omega$
- $\phi(x, t + s) = \phi(\phi(x, t), s)$ *for all* $x \in \Omega$ *and* $t, s \in \mathbb{R}$.

Remark B.2.2. In general the flows we will be interested in will be those that are generated by ordinary differential equations.

Now if we consider the space of continuous function $C(X \times \mathbb{R}, X)$ with the compact open topology then the subset of flows $\mathcal{F}$ is a topological subspace of this space. Furthermore as a flow $\phi \in \mathcal{F}$ is completely determined by

$$\phi_\varepsilon = \phi|_{X\times[-\varepsilon,\varepsilon]},$$

for any $\varepsilon > 0$, the topology on $\mathcal{F}$ corresponds to uniform convergence of the functions ϕ_ε.

Definition B.2.3. *Let $N \subset X$ be the closure of a bounded open set then we define the set of flows for which N is an isolating neighbourhood*

$$\mathcal{U}(N) = \{\phi \in \mathcal{F} : N \text{ is a isolating neighbourhood of } \phi\},$$

which is a open subset of $\mathcal{F}$.

Now if we let $\mathcal{C}(X)$ denote the collection of closed sets of X, then we can define the set of flows with isolating invariant sets.

Definition B.2.4. *The set $\mathcal{S} \subset \mathcal{F} \times \mathcal{C}(X)$ is defined by*

$$\mathcal{S} = \{(\phi, I) : I \text{ is a isolated invariant set of } \phi\}$$

Next we want to put a topology on this set $\mathcal{S}$, which we will then use to define the concept of continuation. Thus, if $N \subset X$ is the closure of a bounded open set in X, we define the function $\sigma_N : \mathcal{U}(N) \to \mathcal{S}$ by

$$\sigma_N(\phi) := (\phi, I)$$

where I is the maximal isolated invariant set of ϕ in N. Using this function we put the topology on $\mathcal{S}$ which is generated by the sets $\phi_N(U)$, where $U \subset \mathcal{U}(N)$ is an open subset of $\mathcal{F}$.

We are now in a position to define the concept of continuation.

Definition B.2.5. *Let $(\phi, I), (\phi', I') \in \mathcal{S}$ then we say that they are related by continuation if they are in the same connected component of $\mathcal{S}$.*

This definition leads to the following important theorem.

Theorem B.2.6. *[45, Theorem 23.31] If $(\phi, I), (\phi', I') \in \mathcal{S}$ are related by continuation then the isolated invariant sets I and I' have the same Conley index.*

Thus we have a condition for two isolated invariant set to have the same Conley index. Hence the problem is now showing that two elements of $\mathcal{S}$ are in the same connected component. Now if we define the projection $\pi : \mathcal{S} \to \mathcal{F}$ by

$$\pi(\phi, I) = \phi,$$

then $\pi \circ \sigma_N : \mathcal{U}(N) \to \mathcal{U}(N)$ and we have the following result which will allow us to test when two elements of $\mathcal{S}$ are in the same connected component.

Theorem B.2.7. *[45, Theorem 22.5] The projection $\pi : \mathcal{S} \to \mathcal{F}$ is a local homeomorphism.*

Remark B.2.8. Thus we can use the following argument. If ϕ and ϕ' are in the same connected component C of $\mathcal{U}(N)$, then by the above theorem $\pi^{-1}(C)$ is a connected component in $\mathcal{S}$ and thus it follows that $\sigma_N(\phi) = (\phi, I)$ and $\sigma_N(\phi') = (\phi', I')$ are related by continuation. Hence we have reduced the problem to showing that two flows are in the connected component of $\mathcal{U}(N)$.

We will now use these results to show that a small perturbation of an ordinary differential equation does not change the Conley index.

Lemma B.2.9. *Let $f, g \in C^1(\mathbb{R}^2)$ and suppose the ordinary differential equation*

$$\dot{x} = f(x) \tag{B.1}$$

has an isolating block $B \subset \mathbb{R}^2$ which contains a maximal isolated invariant set I with Conley index $h(I)$. Then B is an isolating neighbourhood for the ordinary differential equation

$$\dot{x} = f(x) + \delta g(x), \tag{B.2}$$

for $\delta \in \mathbb{R} \setminus \{0\}$ with $|\delta|$ sufficiently small, and the Conley index of the maximal isolated invariant set I_δ contained in B with respect to (B.2) *is equal to $h(I)$.*

Proof: The flows generated by the ordinary differential equations (B.1) and (B.2) are defined on the whole of $\mathbb{R}^2$, so the first step is to restrict these flows to a compact set. Thus let R_1 be a closed bounded rectangle which compactly contains B and R_2 be a seconded closed bounded rectangle which compactly contains R_1. Now let $\chi : \mathbb{R}^2 \to [0,1]$ be a smooth cut-off function such that $\chi(x) = 1$ for all $x \in R_1$ and $\chi(x) = 0$ for all $x \in \mathbb{R}^2 \setminus R_2$. Then we consider the restricted ordinary differential equations

$$\dot{x} = \chi(x) f(x) =: \hat{f}(x) \tag{B.3}$$

and

$$\dot{x} = \chi(x) \left(f(x) + \delta g(x)\right) =: \hat{f}(x) + \delta \hat{g}(x). \tag{B.4}$$

the flows generated by these restricted equations are the same as those generated by (B.1) and (B.2) on R_1 and stop at the boundary of R_2. Thus these two equations generate flows on the rectangle R_2, $\phi : R_2 \times \mathbb{R} \to R_2$ and $\phi_\delta : R_2 \times \mathbb{R} \to R_2$ respectively.

Now we want to show that both ϕ and ϕ_δ are in the same connected component of $\mathcal{U}(B)$ for $|\delta|$ sufficiently small and thus that (ϕ, I) and (ϕ_δ, I_δ) are related by continuation. To achieve this we need to estimate the distance between the flows after a certain time. Thus if we let $\varepsilon > 0$, $p \in \mathbb{R}^2$ and $t \in [-\varepsilon, \varepsilon]$ be arbitrary then it follows from equations (B.3) and (B.4) that

$$\phi(p,t) = p + \int_0^t \hat{f}\left(\phi(p,s)\right) \mathrm{d}s$$

and

$$\phi_\delta(p,t) = p + \int_0^t \hat{f}\left(\phi_\delta(p,s)\right) \mathrm{d}s + \delta \int_0^t g\left(\phi_\delta(p,s)\right) \mathrm{d}s.$$

Therefore, as $\hat{f}$ is Lipschitz continuous with Lipschitz constant $L_{\hat{f}}$ and $\hat{g}$ is bounded with $|\hat{g}(x)| \leqslant M_{\hat{g}}$ for all $x \in \mathbb{R}$, it follows from

$$\phi_\delta(p,t) - \phi(p,t) = \int_0^t \hat{f}\left(\phi_\delta(p,s)\right) - \hat{f}\left(\phi(p,s)\right) \mathrm{d}s + \delta \int_0^t g\left(\phi_\delta(p,s)\right) \mathrm{d}s$$

that

$$\begin{aligned}\max_{t \in [-\varepsilon,\varepsilon]} |\phi_\delta(p,t) - \phi(p,t)| &\leqslant \varepsilon L_{\hat{f}} \max_{t \in [-\varepsilon,\varepsilon]} |\phi_\delta(p,t) - \phi(p,t)| + \varepsilon\delta M_{\hat{g}} \\ &\leqslant \varepsilon\delta M_{\hat{g}} \exp\left(L_{\hat{f}}\varepsilon\right) \to 0 \text{ as } \delta \to 0.\end{aligned}$$

Notice that this estimate is independent of the choice of $p \in \mathbb{R}^2$, so

$$\phi_\delta|_{R_2 \times [-\varepsilon,\varepsilon]} \to \phi|_{R_2 \times [-\varepsilon,\varepsilon]} \text{ as } \delta \to 0$$

uniformly.

Now to show that ϕ and ϕ_δ are in the same connected component of $\mathcal{U}(B)$ we first need to show that B is an isolating neighbourhood for ϕ_δ. Since B is an isolating block for ϕ and thus an isolating neighbourhood it follows that for each $x \in \partial B$ there exist a $t_x \in \mathbb{R}$ such that $\phi(x, t_x) \in R_2 \setminus B$ and since $R_2 \setminus B$ is an open subset of R_2 there exists a $\rho_x > 0$ such that $B_{\rho_x}(\phi(x,t_x), R_2) \cap B = \emptyset$. Now as $\phi(\cdot, t_x) : R_2 \to R_2$ is continuous the set $O_x := \phi^{-1}\left(B_{\frac{\rho_x}{2}}(\phi(x,t_x), R_2), t_x\right)$ is a open neighbourhood of $x \in \partial B$. Hence the collection of open sets

$$\{O_x : x \in \partial B\}$$

is a open cover of the compact set ∂B and thus has a finite subcover $O_{x_1}, \ldots O_{x_n}$. Thus if we let

$$t_{max} = \max\{|t_{x_1}|, \ldots, |t_{x_n}|\}$$

and

$$\rho_{min} = \min\left\{\frac{\rho_{x_1}}{2}, \ldots, \frac{\rho_{x_n}}{2}\right\},$$

then it follows that for any $x \in \partial B$ there exists a $\hat{t}_x \in [-t_{max}, t_{max}]$ such that

$$B_{\rho_{min}}\left(\phi(x, \hat{t}_x), R_2\right) \cap B = \emptyset. \tag{B.5}$$

Now, as

$$\phi_\delta|_{R_2 \times [-\varepsilon, \varepsilon]} \to \phi|_{R_2 \times [-\varepsilon,\varepsilon]} \text{ as } \delta \to 0$$

uniformly for all $\varepsilon > 0$, there will exist a $\Gamma > 0$ such that

$$\max_{p \in R_2} \max_{t \in [-t_{max}, t_{max}]} |\phi_\delta(p,t) - \phi(p,t)| \leqslant \frac{\rho_{min}}{2}$$

for all $\delta \in [-\Gamma, \Gamma]$. Hence by combining this estimate with (B.5) we get that for any $x \in \partial B$ there exists a $\tilde{t}_x \in [-t_{max}, t_{max}]$ such that

$$B_{\frac{\rho_{min}}{2}}\left(\phi_\delta(x, \tilde{t}_x), R_2\right) \cap B = \emptyset.$$

Therefore we have that B is a isolating neighbourhood for ϕ_δ for all $\delta \in [-\Gamma, \Gamma]$. Thus it just remains to show that ϕ and ϕ_δ are in the same connected component of $\mathcal{U}(B)$. To see this notice that the function $\psi : [-\Gamma, \Gamma] \to \mathcal{U}(B)$ defined by

$$\psi(\delta) = \begin{cases} \phi_\delta & \text{for } \delta \neq 0 \\ \phi & \text{for } \delta = 0 \end{cases}$$

is continuous, which can be seen by a similar calculation to the one used to show the uniform convergence of the flows. Therefore the flows ϕ_δ for $\delta \in [-\Gamma, \Gamma]$ are path connected to ϕ and thus are in the same connected component of $\mathcal{U}(B)$. Thus, by remark B.2.8, (ϕ_δ, I_δ) and (ϕ, I) are in the same connected component of $\mathcal{S}$ and therefore by theorem B.2.6 it follows that $h(I) = h(I_\delta)$.

□

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